# 强作用幺正费米气体声速公式的普适性

刘可
liuke@phy.ccnu.edu.cn

华中师范大学物理学院

2010.4.19

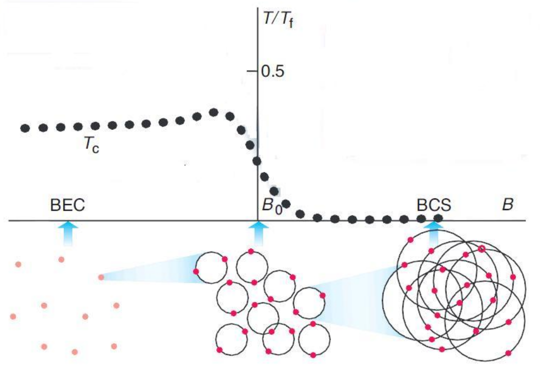
# 主要内容

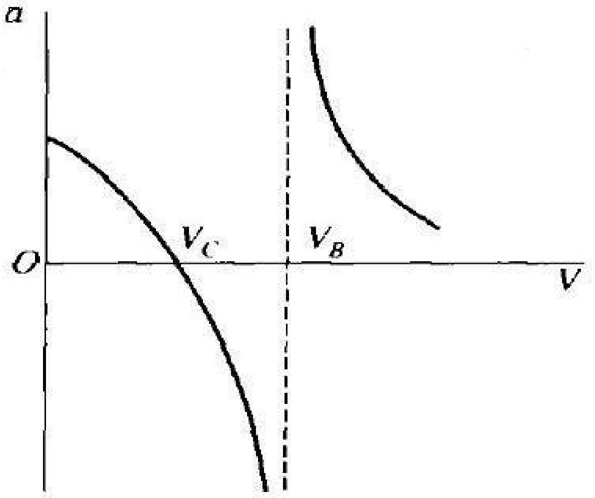
**玻色子——→玻色爱因斯坦凝聚.**

费米子——→费米凝聚？？

**费米子—→（配对）—→玻色子—→凝聚**

# BCS-BEC crossover




a
O
$V_C$
$V_B$
V

### 一般情况

热力学量为$a, T, n$的函数.

### 幺正极限

$|a| = \infty$，显然不能包含于热力学量中.

幺正极限下，系统呈现热力学普适性，
例如$P = \frac{2}{3}\frac{E}{V}$依然成立.

# 声速公式

$$a^2 = \left(\frac{\partial P}{\partial \rho}\right)_S = \frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_S, \tag{1}$$

**注**

$P$为压强, $\rho = mn$为介质的密度, 其中$n = N/V$为粒子数密度.

**问题**

求偏微分$\left(\frac{\partial P}{\partial n}\right)_S$并不是一个简单的任务, 因为压强$P$和粒子数密度$n$是强度量, 而熵$S$是广延量.

# 问题的解决

利用雅可比行列式的性质可以把上面的式子化为

$$\left(\frac{\partial P}{\partial n}\right)_S = \left(\frac{\partial P}{\partial n}\right)_T - \left(\frac{\partial P}{\partial T}\right)_n \left(\frac{\partial (s/n)}{\partial n}\right)_T \Big/ \left(\frac{\partial (s/n)}{\partial T}\right)_n . \qquad (2)$$

**注**

*在上式中，我们把熵$S$变成了熵密度$s = S/V$.*

## 经典波尔兹曼气体的声速公式

对于经典波尔兹曼气体, 压强和熵密度分别为

$$P = nT, s = \frac{S}{V} = n\left\{\ln\left[\frac{1}{n}\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}}\right] + \frac{5}{2}\right\}.$$

$$\left(\frac{\partial P}{\partial n}\right)_T = T, \left(\frac{\partial(s/n)}{\partial n}\right)_T = -n^{-1}, \left(\frac{\partial P}{\partial T}\right)_n = n, \left(\frac{\partial(s/n)}{\partial T}\right)_n = \frac{3}{2}T^{-1}.$$

**代入Eq.(2)可得**

$$a = \sqrt{5T/(3m)} = \boxed{\sqrt{\frac{5}{3}\frac{P}{mn}}.} \tag{3}$$

# 理想玻色气体

当温度低于临界温度$T_C$($T_C = \frac{2\pi}{m}\left(\frac{n}{\zeta[\frac{3}{2}]}\right)^{2/3}$)时，会出现玻色爱因斯坦凝聚(BEC)现象. 所以，对于理想玻色气体的声速公式需要在$T < T_C$ 和$T > T_C$两种情况下分别讨论.

## 当$T > T_C$时，压强、粒子数密度和熵密度分别为

$$P = \frac{T}{\lambda^3} g_{5/2}(z), s = n\left[\frac{5}{2}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \ln z\right], n = \frac{1}{\lambda^3} g_{3/2}(z).$$

## 注

$\lambda = \sqrt{2\pi/mT}$为德布罗意波长，$g_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}dx}{z^{-1}e^x - 1}$，
$z = e^{\mu/T}$为易逸度.

# 理想玻色气体

利用函数$g_\nu(z)$的性质$z\partial/\partial z[g_\nu(z)] = g_{\nu-1}(z)$，可以得到

$$\left(\frac{\partial P}{\partial n}\right)_T = \frac{nT}{z}\left(\frac{\partial z}{\partial n}\right)_T, \left(\frac{\partial P}{\partial T}\right)_n = \frac{5}{2}\frac{}{\lambda^3}g_{5/2}(z) + \frac{nT}{z}\left(\frac{\partial z}{\partial T}\right)_n,$$

$$\left(\frac{\partial(s/n)}{\partial n}\right)_T = \left[-\frac{5}{2n}\frac{g_{5/2}(z)}{g_{3/2}(z)} + \frac{3}{2z}\left(\frac{\partial z}{\partial n}\right)_T\right],$$

$$\left(\frac{\partial(s/n)}{\partial T}\right)_n = \frac{3}{2}\left[\frac{5}{2T}\frac{g_{5/2}(z)}{g_{3/2}(z)} + \frac{1}{z}\left(\frac{\partial z}{\partial T}\right)_n\right].$$

代入Eq.(2)可得

$$a = \boxed{\sqrt{\frac{5}{3}\frac{P}{mn}}}. \tag{4}$$

## 理想玻色气体

**当 $T < T_C$时, $z = e^{\mu/T} = 1$, 压强和熵密度则分别为**

$$P = \frac{T}{\lambda^3} g_{5/2}(1), \ s = \frac{S}{V} = \frac{5}{2}\frac{1}{\lambda^3} g_{5/2}(1).$$

于是,

$$\left(\frac{\partial P}{\partial n}\right)_T = 0, \ \left(\frac{\partial P}{\partial T}\right)_n = \frac{5}{2}\frac{1}{\lambda^3} g_{5/2}(1),$$

$$\left(\frac{\partial (s/n)}{\partial n}\right)_T = -\frac{5}{2}\frac{1}{n^2\lambda^3} g_{5/2}(1),$$

$$\left(\frac{\partial (s/n)}{\partial T}\right)_n = \frac{15}{4}\frac{1}{n\lambda^3 T} g_{5/2}(1),$$

## 理想玻色气体

**代入Eq.(2)可得**

$$a = \sqrt{\frac{5}{3}\frac{T}{mn\lambda^3}g_{5/2}(1)} = \boxed{\sqrt{\frac{5}{3}\frac{P}{mn}}}. \tag{5}$$

可以看到，在$T < T_C$ 和$T > T_C$的范围内，理想玻色气体的声速公式同经典玻尔兹曼气体的相同.

# 理想费米气体

## 理想费米气体的压强、粒子数密度和熵密度的表达式分别为

$$
\begin{aligned}
P &= \frac{2T}{\lambda^3} f_{5/2}(z), \\
s &= \frac{S}{V} = n\left[\frac{5}{2}\frac{f_{5/2}(z)}{f_{3/2}(z)} - \ln z\right], \\
n &= \frac{2}{\lambda^3} f_{3/2}(z).
\end{aligned}
$$

## 注

*在上面的式子中，* $f_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}dx}{z^{-1}e^x+1}$ *为费米积分，且有如下性质* $z\partial/\partial z[f_\nu(z)] = f_{\nu-1}(z)$.

# 理想费米气体

通过类似于理想玻色气体$T > T_C$时的方法，可以很快得到理想费米气体的声速公式，

**即**

$$a = \sqrt{\frac{10T}{3m}\frac{f_{5/2}(z)}{f_{3/2}(z)}} = \boxed{\sqrt{\frac{5}{3}\frac{P}{mn}}}. \tag{6}$$

从以上的分析可以看出，声速公式对于量子和经典的理想气体是相同的.

## 幺正费米气体

在之前的工作中，我们利用准线性近似方法导出了幺正费米气体的一些热力学量( Phys. Rev. **A 79**, 043625 (2009)，Phys. Rev. **A 76**, 033617 (2007)).

压强$P$和单粒子熵密度分别为

$$P = \frac{2T}{\lambda^3}\left(f_{5/2}(z^{'}) - \frac{f_{3/2}^2(z^{'})}{2f_{1/2}(z^{'})} + \frac{f_{3/2}^3(z^{'})f_{-1/2}(z^{'})}{2f_{1/2}^3(z^{'})}\right),$$

$$\frac{s}{n} = \frac{5}{2}\frac{f_{5/2}(z^{'})}{f_{3/2}(z^{'})} - \ln z^{'} + \frac{3f_{-1/2}(z^{'})f_{3/2}^2(z^{'})}{4f_{1/2}^3(z^{'})} - \frac{f_{3/2}(z^{'})}{4f_{1/2}(z^{'})},$$

$z^{'} = e^{\mu^*/T}$ 为有效易逸度.

## 幺正费米气体

为了简化计算，令$P^* = f_{5/2}(z') - \frac{f_{3/2}^2(z')}{2f_{1/2}(z')} + \frac{f_{3/2}^3(z')f_{-1/2}(z')}{2f_{1/2}^3(z')}$.

则有，

$$\left(\frac{\partial P}{\partial T}\right)_n = \frac{5}{\lambda^3}P^* + \frac{2T}{\lambda^3}\frac{\mathrm{d}P^*}{\mathrm{d}z'}\left(\frac{\partial z'}{\partial T}\right)_n,$$

$$\left(\frac{\partial P}{\partial n}\right)_T = \frac{2T}{\lambda^3}\frac{\mathrm{d}P^*}{\mathrm{d}z'}\left(\frac{\partial z'}{\partial n}\right)_T,$$

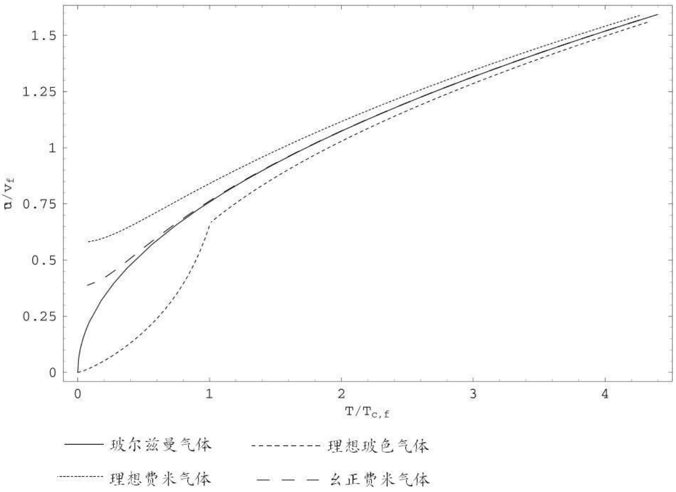
$$\left(\frac{\partial(s/n)}{\partial T}\right)_n = \frac{\mathrm{d}(s/n)}{\mathrm{d}z'}\left(\frac{\partial z'}{\partial T}\right)_n,$$

$$\left(\frac{\partial(s/n)}{\partial n}\right)_T = \frac{\mathrm{d}(s/n)}{\mathrm{d}z'}\left(\frac{\partial z'}{\partial n}\right)_T.$$

# 幺正费米气体

代入Eq.(2),并利用函数关系$\left(\frac{\partial z'}{\partial n}\right)_T\left(\frac{\partial T}{\partial z'}\right)_n\left(\frac{\partial n}{\partial T}\right)_{z'}=-1$, 就可以得到幺正费米气体的声速公式

$$a=\sqrt{\left(\frac{\partial P}{\partial n}\right)_S}=\sqrt{\frac{10}{3}\frac{T}{n\lambda^3}P^*}=\boxed{\sqrt{\frac{5}{3}\frac{P}{mn}}}. \tag{7}$$


1.5
1.25
1
0.75
0.5
0.25
0
u/v_f
0
1
2
3
4
T/T_c,f
玻尔兹曼气体
理想玻色气体
理想费米气体
幺正费米气体

# 小结

幺正费米气体的声速公式和理想气体的相同，都是$\boxed{\sqrt{\frac{5}{3}\frac{P}{mn}}.}$

## 谢谢