NNLO corrections to rare and hadronic B-meson decays

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Overview of rare and hadronic B-meson decays

- B-meson weak decays play a very important role in:
  - testing the Standard Model;
  - probing the origin of CP violation;
  - searching for indirect signals of New Physics.

- Exp. side: more data and more precise due to:
  - BaBar at SLAC and Belle at KEK;
  - Tevatron at Fermilab and LHC-b at CERN;
  - higher luminosity Super-B factory.

- Theo. side: various theoretical frameworks proposed:
  - methods based on flavour symmetries of QCD;
  - methods based on factorization theorems of QCD dynamics: PQCD, QCDF and SCET;
  - non-perturbative methods, Lattice QCD or QCDSR;
B physics at the NNLO frontier: current status

- NNLO program for $\mathcal{H}_{\text{eff}} = \sum_i C_i Q_i$ now complete:
  - 2-loop/3-loop matching corrections [Bobeth, Misiak, Urban 99; Misiak, Steinhauser 04]
  - 3-loop/4-loop anomalous dimensions [Gorbahn, Haisch 04; Gorbahn, Haisch, Misiak 05; Czakon, Haisch, Misiak 06]

⇒ need to calculate the hadronic matrix elements to the same level of precision!

- Some NNLO analysis of B-meson decays:
  - $B \rightarrow X_s \gamma$ [M. Misiak et al. 06; Becher, Neubert 06]
  - $B \rightarrow X_u \ell \nu$ [Greub, Neubert, Pecjak 09]
  - $B \rightarrow M_1 M_2$ [Beneke, Jaeger 05,06; Bell 07,09; Jain, Rothstein, Stewart 07; Beneke, Huber, X. Q. Li 09; X. Q. Li, Y. D. Yang 05,06]
The effective weak Hamiltonian

- Effective weak Hamiltonian: [BBL basis Buras, Buchalla, Lautenbacher’96; CMM basis Chetyrkin, Misiak, Münz’98]

\[ H_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_{p=u,c} \lambda_p \left[ C_1 Q_1^p + C_2 Q_2^p + \sum_{k=3}^6 C_k Q_k + C_7 Q_7 + C_8 Q_8 \right] + \text{h.c.} \]

- \( C_i \): include physics from \( m_W \) down to \( m_b \) scale; perturbatively calculable; have been calculated to NNLO [Gorbahn and Haisch 04].

- Adopting CMM basis: convenient for multi-loop calculation; can safely use NDR scheme with anti-commuting \( \gamma_5 \).

- In NDR scheme, needs include evanescent operators. [Gorbahn and Haisch 04].
  - vanishing in 4 dim., important in the intermediate step,
  - needed to complete the operator basis under renormalization
QCD factorization approach

BBNS factorization formula:

\[ \langle M_1 M_2 | Q_i | \bar{B} \rangle \simeq F^{BM_1}_+ (0) f_{M_2} \int du \ T^I_{i}(u) \ \phi_{M_2}(u) \]

\[ + \hat{f}_B f_{M_1} f_{M_2} \int d\omega dv du \ T^{II}_{i}(\omega, v, u) \ \phi_B(\omega) \ \phi_{M_1}(v) \ \phi_{M_2}(u) \]

Perturbative calculation of hard scattering kernels $T^{I,\, II}$

- $T^{I,\, II}$: perturbative calculable order by order in $\alpha_s$;
  
  \[
  T^I = \mathcal{O}(1), \quad \text{while} \quad T^{II} = \mathcal{O}(\alpha_s).
  \]

- Relevant Feynman diagram at NLO in $\alpha_s$:

- At NLO, all the relevant 130 two-body charmless decay modes have been analyzed:
  
  \text{BBNS; BN’03; Du; Cheng, for } B \rightarrow PP, PV; \quad \text{BRY’07; Kagan; Li and Yang; Cheng and Yang, for } B \rightarrow VV; \quad \ldots
Factorization formulae for $B \to M_1 M_2$ in SCET, I

- Soft-collinear effective theory:
  - an EFT describing energetic ($E \gg \Lambda_{QCD}$) hadrons/jets;
  - suitable for studying factorization, resummation and power corrections;
  - successfully applied to $B \to X_s \gamma$, $B \to D \pi$, $B \to X_u \ell \nu$, $B \to X_s \ell^+ \ell^-$;

- two different formulations:
  1. SCET in momentum space, Hybrid expanded:
     C. W. Bauer, S. Fleming, D. Pirjol and I. W. Stewart, 00
     C. W. Bauer, D. Pirjol and I. W. Stewart, 01
  2. SCET in position space, multi-pole expanded:
     M. Beneke, A. P. Chapovsky, M. Diehl and Th. Feldmann, 02
     M. Beneke and Th. Feldmann, 02
     R. J. Hill and M. Neubert, 02

- Both are equivalent at all order, but do not coincide order by order in $\lambda$. 
Factorization formulae for $B \to M_1 M_2$ in SCET, II

- $T_I, II$: more transparent in SCET; just operator matching calculations; matching coefficients of some SCET (non) local operators;

- Matching procedure for $T_{II}$: complicated due to $\mu_{hc}$

\[
\text{QCD} \to \text{SCET}_I(hc, c, s) \to \text{SCET}_{II}(c, s)
\]

- Final result for $T_{II}^I$: $T_{II}^I(\omega, \nu, u) = \int dz J(\omega, \nu, z) H_i(z, u)$
  - $H_i = \mathcal{O}(1)$: hard coefficient, QCD $\to$ SCET$_I$ at $\mu_h \sim m_b$;
  - $J = \mathcal{O}(\alpha_s)$: jet function, SCET$_I$ $\to$ SCET$_{II}$ at $\mu_{hc} \sim \sqrt{m_b \Lambda_{QCD}}$;
  - resummation of $\log \mu_h/\mu_{hc}$ using RGEs in SCET;  

[Beke, Yang 05]
Factorization formula for $B \to M_1 M_2$ in SCET, III

- Matching procedure for $T^I$: conceptually simpler, no intermediate scale $\mu_{hc}$;

  \[
  \text{QCD} \to \text{SCET}_1(hc, c, s)
  \]

- Final result for the hard kernel $T^I_i$:

  \[
  T^I_i(u) = T^I_i^{(0)}(u) + \alpha_s T^I_i^{(1)}(u) + \alpha_s^2 T^I_i^{(2)}(u)
  \]

  - $T^I_i^{(0)}$: naive factorization
  - $T^I_i^{(1)}$: NLO vertex correction
  - $T^I_i^{(2)}$: NNLO vertex correction, true 2-loop calculation.
Motivation for NNLO calculation

- Final results for $T_{i}^{I,II}$:

$$T_{i}^{I}(u) = T_{i}^{I(0)}(u) + \alpha_s T_{i}^{I(1)}(u) + \alpha_s^2 T_{i}^{I(2)}(u)$$

$$T_{i}^{II}(\omega, v, u) = \int dz J(\omega, v, z) H_i(z, u)$$

- Phenomenologically very relevant:
  - only the first corrections to strong phases, quite relevant to direct $CP$;
  - needed to reduce the large (N)LO scale uncertainties, generally expected;

- Current data driven: $C/T$ or $\alpha_2$ seems to be too small, large cancelation in LO + NLO, particularly sensitive to NNLO, $\implies$ enhancement from NNLO?

- Conceptual and systematic aspects:
  - verification of factorization at NNLO, still hold at NNLO?
  - spectator scatterings involve $\mu_{hc}$, PT well-behaved?
Status of NNLO calculation

- Define the topological amplitudes:

  - colour-allowed tree $\alpha_1$
  - colour-suppressed tree $\alpha_2$
  - QCD penguins $\alpha_4$

- Available NNLO corrections:
  - $J$: matching+resummation,
    
    $[\text{Beneke,Yang 05; Becher,Hill,Lee,Neubert 04; Kirilin 05}]$

  - $H_i$: for tree amplitudes,
    for penguin amplitudes,
    
    $[\text{Beneke,Jaeger 05; Kivel 06; Pilipp 07}]
    [\text{Beneke,Jaeger 06; Li and Yang 06; Jain,Rothstein,Stewart 07}]$

  - $T^I$: for tree amplitudes,
    for penguin amplitudes,
    
    $[\text{Bell 06 (Im part), 09 (Re part); Beneke,Huber,Li, 09}]
    [\text{more complicated, to be done next!}]$
NNLO vertex corrections to $T^I$

- due to the absence of the intermediate scale $\mu_{hc}$, conceptually simpler;
- technically demanding, a genuine 2-loop calculation with four external legs.

- direct diagrammatical approach \([G.\text{Bell 07, 09}] \Leftrightarrow \) matching calculation within SCET \([M. \text{Beneke, T. Huber, X.Q. Li 09}] \Rightarrow \text{an important cross-check!}\)

- two different contraction of current-current operators:

![Diagram of two different current-current operators](image)

- two main tasks: calculating 2-loop QCD Feynman diagrams, performing complicated IR-subtraction.
Two-loop Feynman diagrams

- Two-loop non-factorizable diagrams:

- totally 62 “non-factorizable” diagrams;
- vacuum polarization insertions in gluon propagators;
- external self-energy insertions in quark propagators;
- one-loop counter-term insertions;
Multi-loop calculations in a nutshell

- Work in DR with $D = 4 - 2\epsilon$, to regulate both UV and IR div.; at 2-loop order, IR poles appear up to $1/\epsilon^4$.

- Basis strategy:
  - general tensor decomposition via Passarino-Veltman ansatz, $\Rightarrow$ thousands of scalar integrals, \[ \text{[Passarino, Veltman'79]}; \]
  - reduction them to Master Integral via Laporta algorithm based on Integration-by-part (IBP) identities $\Rightarrow$ totally 42 MIs, \[ \text{[Tkachov'81; Chetyrkin, Tkachov'81]} \]
    \[ \text{[Laporta'01; Anastasiou, Lazopoulos'04; Smirnov'08]}; \]
  - calculation of the MIs, very challenging, all should be given analytically.

- Techniques used to calculate MIs:
  - standard Feynman parameterisation, only for simpler MIs; \[ \text{[Kotikov'91; Remiddi'97]}; \]
  - method of differential equations, \[ \text{[Smirnov'99; Tausk'99]}; \]
  - Mellin-Barnes techniques,
Master formula for hard scattering kernel in RI

\[
T_i^{(1)} = A_{i1}^{(1),nf} + Z_{ij}^{(1)} A_{j1}^{(0)},
\]

\[
T_i^{(2)} = A_{i1}^{(2),nf} + Z_{ij}^{(1)} A_{j1}^{(1)} + Z_{ij}^{(2)} A_{j1}^{(0)} + Z_{ij}^{(1)} A_{i1}^{(1),nf} + (-i) \delta_{m}^{(1)} A'_{i1}^{(1),nf} \\
+ T_i^{(1)} \left[ - C_{FF}^{(1)} - Y_{11} + Z_{ext}^{(1)} \right] - \sum_{b>1} H_{ib}^{(1)} Y_{b1}^{(1)}.
\]

- \( T_i^{(1)} \) and \( T_i^{(2)} \): 1-loop and 2-loop hard scattering kernel;
- important check: both are free of poles in \( \epsilon \), factorization holds.
- needs higher-order \( \epsilon \) terms in \( T_i^{(1)} \) to compute \( T_i^{(2)} \);
- \( Z_{ij} \): renormalisation constants of QCD operators;
- \( Y_{ij} \): renormalization constants of the SCET operators;
- \( C_{FF} \): matching coefficient of QCD current to SCET current;
- needs also 1-loop factorizable diagrams;
Master formula for hard scattering kernel in WI

\[
\begin{align*}
\tilde{T}^{(1)}_i &= \tilde{A}^{(1),nf}_i + Z_{ij} \tilde{A}^{(0)}_{jl} + \left[ \tilde{A}^{(1),f}_{il} - A^{(1),f}_{l1} \right] - \left[ \tilde{Y}^{(1)}_{11} - Y^{(1)}_{11} \right] \tilde{A}^{(0)}_{il} \\
\tilde{T}^{(2)}_i &= \tilde{A}^{(2),nf}_i + Z_{ij} \tilde{A}^{(1)}_{jl} + Z^{(1)}_{ij} \tilde{A}^{(0)}_{j1} + Z^{(1)}_{ij} \tilde{A}^{(1),nf}_{jl} \\
&\quad + (-i) \delta^{(1)}_{m} \tilde{A}^{(1),nf}_{i1} + Z^{(1)}_{ext} \left[ \tilde{A}^{(1),nf}_{i1} + Z^{(1)}_{ij} \tilde{A}^{(1)}_{jl} \right] \\
&\quad - \tilde{T}^{(1)}_{i1} \left[ C_{FF}^{(1)} + \tilde{Y}^{(1)}_{11} \right] - \sum_{b>1} H^{(1)}_{ib} Y^{(1)}_{b1} \\
&\quad + \left[ \tilde{A}^{(2),f}_{i1} - A^{(2),f}_{l1} \tilde{A}^{(0)}_{l1} \right] + (-i) \delta^{(1)}_{m} \left[ \tilde{A}^{(1),f}_{i1} - A^{(1),f}_{l1} \tilde{A}^{(0)}_{l1} \right] \\
&\quad + \left( Z^{(1)}_{\alpha} + Z^{(1)}_{ext} + \xi^{(1)}_{45} \right) \left[ \tilde{A}^{(1),f}_{i1} - A^{(1),f}_{l1} \tilde{A}^{(0)}_{l1} \right] \\
&\quad - C_{FF}^{(1)} \tilde{A}^{(0)}_{i1} \left[ \tilde{Y}^{(1)}_{11} - Y^{(1)}_{11} \right] - \left[ \tilde{Y}^{(2)}_{11} - Y^{(2)}_{11} \right] \tilde{A}^{(0)}_{i1}
\end{align*}
\]

- Fierz\(\tilde{O}_1\)=\(O_1\) in \(D = 4\) dim., so \(\tilde{O}_1 - O_1\) is also evanescent;
Dependence of $\alpha_{1,2}$ on the hard scale $\mu_h$, only vertex part!

- dotted line: LO result
- dashed line: NLO result
- solid line: NNLO result

- the real parts on the scale dependence substantially reduced!
- the imaginary parts less pronounced, since the NNLO term is really NLO!
Numerical result for $\alpha_1$ and $\alpha_2$

\[ \alpha_1(\pi\pi) = 1.009 + [0.023 + 0.010i]_{\text{NLO}} + [0.026 + 0.028i]_{\text{NNLO}} - \left[ \frac{r_{sp}}{0.445} \right] \left\{ [0.014]_{\text{LOsp}} + [0.034 + 0.027i]_{\text{NLOsp}} + [0.008]_{\text{tw3}} \right\} \]
\[ = 1.000^{+0.029}_{-0.069} + (0.011^{+0.023}_{-0.050})i \]

\[ \alpha_2(\pi\pi) = 0.220 - [0.179 + 0.077i]_{\text{NLO}} - [0.031 + 0.050i]_{\text{NNLO}} + \left[ \frac{r_{sp}}{0.445} \right] \left\{ [0.114]_{\text{LOsp}} + [0.049 + 0.051i]_{\text{NLOsp}} + [0.067]_{\text{tw3}} \right\} \]
\[ = 0.240^{+0.217}_{-0.125} + (-0.077^{+0.115}_{-0.078})i \]

- The NNLO corrections to the vertex term and spectator scattering are significant individually. But both tend to cancel each other. too bad!

- Largest uncertainty from $r_{sp} = \frac{g f_{\pi} \hat{f}_B}{m_b f^B_{+}(0) \lambda_B}$, especially $\lambda_B$ (B LCDA).

- Perturbation theory works at scales $m_b$ and $\mu_{hc} = \sqrt{\Lambda_{\text{QCD}} m_b}$. 
Factorization test with sem-leptonic data

\[ R_\pi \equiv \frac{\Gamma(B^- \to \pi^-\pi^0)}{d\Gamma(\bar{B}^0 \to \pi^0l^-\bar{\nu})/dq^2 \bigg|_{q^2=0}} = 3\pi^2 f_\pi^2 |V_{ud}|^2 |\alpha_1(\pi\pi) + \alpha_2(\pi\pi)|^2 \]

- From sem-leptonic data
  \[ |\alpha_1(\pi\pi) + \alpha_2(\pi\pi)|_{\text{exp}} = 1.29 \pm 0.11 \]

- Prediction with \( \lambda_B = 0.35 \text{ GeV} \):
  \[ |\alpha_1(\pi\pi) + \alpha_2(\pi\pi)| = 1.24^{+0.16}_{-0.10} \]

- Good agreements observed, supporting QCD factorization.

- Main uncertainties:
  \( \lambda_B, \alpha_2^\pi \) and power corrections.

- Interesting to extend to other final state; \( R_\rho = 1.75^{+0.37}_{-0.24} (2.08^{+0.50}_{-0.46}); \)
Branching ratios for tree-dominated $B$ decays

<table>
<thead>
<tr>
<th>Decay</th>
<th>Theory I</th>
<th>Theory II</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B^- \rightarrow \pi^- \pi^0$</td>
<td>5.43 $^{+0.06}<em>{-0.06}$ $^{+1.45}</em>{-0.84}$</td>
<td>5.82 $^{+0.07}<em>{-0.06}$ $^{+1.42}</em>{-1.35}$</td>
<td>5.59 $^{+0.41}_{-0.40}$</td>
</tr>
<tr>
<td>$\bar{B}_d^0 \rightarrow \pi^+ \pi^-$</td>
<td>7.37 $^{+0.86}<em>{-0.69}$ $^{+1.22}</em>{-0.97}$</td>
<td>5.70 $^{+0.70}<em>{-0.55}$ $^{+1.16}</em>{-0.97}$</td>
<td>5.16 ± 0.22</td>
</tr>
<tr>
<td>$\bar{B}_d^0 \rightarrow \pi^0 \pi^0$</td>
<td>0.33 $^{+0.11}<em>{-0.08}$ $^{+0.42}</em>{-0.17}$</td>
<td>0.63 $^{+0.12}<em>{-0.10}$ $^{+0.64}</em>{-0.42}$</td>
<td>1.55 ± 0.19</td>
</tr>
<tr>
<td>$B^- \rightarrow \pi^- \rho^0$</td>
<td>8.68 $^{+0.42}<em>{-0.41}$ $^{+2.71}</em>{-1.56}$</td>
<td>9.84 $^{+0.41}<em>{-0.40}$ $^{+2.54}</em>{-2.52}$</td>
<td>8.3 $^{+1.2}_{-1.3}$</td>
</tr>
<tr>
<td>$B^- \rightarrow \pi^0 \rho^-$</td>
<td>12.38 $^{+0.90}<em>{-0.77}$ $^{+2.18}</em>{-1.41}$</td>
<td>12.13 $^{+0.85}<em>{-0.73}$ $^{+2.23}</em>{-1.7}$</td>
<td>10.9 $^{+1.4}_{-1.5}$</td>
</tr>
<tr>
<td>$\bar{B}_d^0 \rightarrow \pi^+ \rho^-$</td>
<td>17.80 $^{+0.62}<em>{-0.56}$ $^{+1.76}</em>{-2.10}$</td>
<td>13.76 $^{+0.49}<em>{-0.44}$ $^{+1.77}</em>{-2.18}$</td>
<td>15.7 ± 1.8</td>
</tr>
<tr>
<td>$\bar{B}_d^0 \rightarrow \pi^- \rho^+$</td>
<td>10.28 $^{+0.39}<em>{-0.39}$ $^{+1.37}</em>{-1.42}$</td>
<td>8.14 $^{+0.34}<em>{-0.33}$ $^{+1.35}</em>{-1.49}$</td>
<td>7.3 ± 1.2</td>
</tr>
<tr>
<td>$\bar{B}_d^0 \rightarrow \pi^\pm \rho^\mp$</td>
<td>28.08 $^{+0.27}<em>{-0.19}$ $^{+3.82}</em>{-3.50}$</td>
<td>21.90 $^{+0.20}<em>{-0.12}$ $^{+3.06}</em>{-3.55}$</td>
<td>23.0 ± 2.3</td>
</tr>
<tr>
<td>$\bar{B}_d^0 \rightarrow \pi^0 \rho^0$</td>
<td>0.52 $^{+0.04}<em>{-0.03}$ $^{+1.11}</em>{-0.43}$</td>
<td>1.49 $^{+0.07}<em>{-0.07}$ $^{+1.77}</em>{-1.29}$</td>
<td>2.0 ± 0.5</td>
</tr>
<tr>
<td>$B^- \rightarrow \rho^- \rho_L$</td>
<td>18.42 $^{+0.23}<em>{-0.21}$ $^{+3.92}</em>{-2.55}$</td>
<td>19.06 $^{+0.24}<em>{-0.22}$ $^{+4.59}</em>{-4.22}$</td>
<td>22.8 $^{+1.8}_{-1.9}$</td>
</tr>
<tr>
<td>$\bar{B}_d^0 \rightarrow \rho^- \rho_L$</td>
<td>25.98 $^{+0.85}<em>{-0.77}$ $^{+2.93}</em>{-3.43}$</td>
<td>20.66 $^{+0.68}<em>{-0.62}$ $^{+2.99}</em>{-3.75}$</td>
<td>23.7 $^{+3.1}_{-3.2}$</td>
</tr>
<tr>
<td>$\bar{B}_d^0 \rightarrow \rho^0 \rho_L$</td>
<td>0.39 $^{+0.03}<em>{-0.03}$ $^{+0.83}</em>{-0.36}$</td>
<td>1.05 $^{+0.05}<em>{-0.04}$ $^{+1.62}</em>{-1.04}$</td>
<td>0.55 $^{+0.22}_{-0.24}$</td>
</tr>
</tbody>
</table>

$f_{B\pi}^+ (0) = 0.23 \pm 0.03$, $A_0^{B\rho} (0) = 0.28 \pm 0.03$, $\lambda_B (1 \text{ GeV}) = (0.20^{+0.05}_{-0.00}) \text{ GeV}$. 
Color-suppressed versus color-allowed decays, I

\[ R_{\pi\pi}^{+} = \frac{2\frac{\Gamma(B^{-} \to \pi^-\pi^0)}{\Gamma(B^0 \to \pi^+\pi^-)}}{\lambda_B[GeV]} \]

- NF fails to describe the data obviously;
- QCDF could describe the data, especially with a smaller \( \lambda_B \);
Color-suppressed versus color-allowed decays, II

\[ R_{00}^{\pi\pi} \equiv 2 \frac{\Gamma(B^0 \rightarrow \pi^0 \pi^0)}{\Gamma(B^0 \rightarrow \pi^+ \pi^-)}. \]

- prefer to smaller $\lambda_B$.
- strong spectator-scattering effects!
Charged $\pi^\mp \rho^\pm$ decay modes

\[ R_3 \equiv \frac{\Gamma(\bar{B}^0 \to \pi^+ \rho^-)}{\Gamma(\bar{B}^0 \to \pi^- \rho^+)} , \]

\[ \Delta C \equiv \frac{1}{2} \left[ C(\pi^- \rho^+) - C(\pi^+ \rho^-) \right] , \]

- Favor slightly smaller $B \to \rho$ to $B \to \pi$ form factor ratio than QCDSR ($\approx 1.25$).
Motivation for the NNLO matching calculation

\[ \bar{q} \Gamma_i b, \text{ with } \Gamma_i = \{1, \gamma_5, \gamma^\mu, \gamma_5 \gamma^\mu, i\sigma^{\mu\nu}\}, \text{ play an important role in B physics:} \]

- govern the hadronic dynamics in inclusive semi-leptonic and radiative B decays: 
  \[ B \rightarrow X_s \gamma, \quad B \rightarrow X_u \ell \nu, \quad B \rightarrow X_s \ell^+ \ell^-, \ldots \]

- its matrix elements \(\Rightarrow\) form factors, also important inputs to exclusive B decay processes;

- in the kinematic region: the FS has small invariant mass but large energy 
  \(\Rightarrow\) SCET is the appropriate theoretical framework, transparent factorization formulae can be derived;

\(\Rightarrow\) how to accurately represent the heavy-to-light currents in SCET is of particular interest!
Heavy-to-light currents in SCET

2-body operators

\[ \bar{q} \Gamma b = \sum_i \int ds \, \tilde{C}^{(A)}_i(s) \, O^{(A)}_i(s) + \sum_i \int ds_1 ds_2 \, \tilde{C}^{(B)}_i(s_1, s_2) \, O^{(B)}_i(s_1, s_2) + \ldots \]

3-body operators

- \( C^{(A)}_i \) and \( C^{(B)}_i \): hard coefficients;
- 1-loop result is well-known,
  \[ [\text{Bauer,Fleming,Pirjol,Stewart 00, Beneke,Kiyo,Yang 04; Becher, Hill 04}] \]
- 2-loop result: known only for (V-A) current,
  \[ [\text{Bell’07,’08; Bonciani,Ferroglia’08; Asatrian,Greub,Pecjak’08; Beneke,Huber,Li’08}] \]

- Inclusive processes: \( \langle O^{(A)}_i \rangle \rightarrow J \otimes S \), \( \langle O^{(B)}_i \rangle \rightarrow \sum_i j_i \otimes s_i \);

- Exclusive processes: \( \langle O^{(A)}_i \rangle \rightarrow \text{soft-overlap contribution} \), \( \langle O^{(B)}_i \rangle \rightarrow \text{hard spectator-scattering} \).
in the leading power:

\[
[\bar{q} \Gamma_i b](0) \simeq \sum_j \int ds \, \tilde{C}_i^j(s) \left[ \bar{\xi} W_{hc} \right] (sn_+) \Gamma_j' h_v(0)
\]

here there are less diagrams and no evanescent operators;
UV renormalization and IR subtraction

- for UV renormalization, needs standard QCD counterterm:
  - the heavy quark mass;
  - the heavy and light quark field; \[ \text{[Gray, Broadhurst, Grafe, Schilcher 90]} \]
  - the QCD current renormalization constant; \[ \text{[Tarrach 81; Broadhurst, Grozin 94]} \]

- for IR subtraction, remember \( C_i C_j \ast J \otimes S \) should be finite:
  - needs the SCET current renormalization constant;
  - 1-loop result known; \[ \text{[Bauer, Fleming, Pirjol, Stewart 00]} \]
  - 2-loop extracted from \( J \otimes S \); \[ \text{[Becher, Neubert 05,06]} \]

\[
Z_J = 1 + \frac{\alpha_s^{(4)} C_F}{4\pi} \left\{ -\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left[ -\ln \left( \frac{\mu^2}{u^2m_b^2} \right) - \frac{5}{2} \right] \right\} + \mathcal{O}(\alpha_s^2)
\]
Matching coefficients $C^j_i$ for various Dirac structures

<table>
<thead>
<tr>
<th>$\Gamma_i$</th>
<th>1</th>
<th>$\gamma_5$</th>
<th>$\gamma^\mu$</th>
<th>$\gamma^5 \gamma^\mu$</th>
<th>$i\sigma^{\mu\nu}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma'_j$</td>
<td>1</td>
<td>$\gamma_5$</td>
<td>$\gamma^\mu$</td>
<td>$\nu^\mu$</td>
<td>$n^\mu_-$</td>
</tr>
<tr>
<td>$C^j_i$</td>
<td>$C_S$</td>
<td>$C_P$</td>
<td>$C_V^1$</td>
<td>$C_V^2$</td>
<td>$C_V^3$</td>
</tr>
</tbody>
</table>

- $\Gamma = \gamma^\mu$: relevant to $B \to X_u \ell \nu_\ell$, $B \to \pi \ell \nu_\ell$;
- $\Gamma = 1, \sigma^{\mu\nu}$: relevant to $B \to X_s \ell^+ \ell^-$, $B \to K(\ast) \ell^+ \ell^-$;

many applications to B decays:
- $V_{ub}$ determination from $B \to X_u \ell \nu_\ell$, [Greub, Neubert and Pecjak 09]
- transition form factor ratios, [Beneke and Feldmann 00, 03; Beneke and Yang 05]
- FBA in inclusive and exclusive $B \to X_s \ell^+ \ell^-$ decays, [Lee, Ligeti, Stewart and Tackmann 06,07; Lee, Ligeti, Tackmann 08]

- details to be found in

Bell, Beneke, Huber, Li, “Matching heavy-to-light current to NNLO in SCET”
Overview of current NNLO corrections to hadronic B decays:
- 1-loop spectator-scattering now complete;
- 2-loop vertex corrections to topological tree amplitudes almost complete;

Factorization shown to be hold in a very non-trivial way:
- both UV- and IR- divergences cancel;
- convolution integrations with LCDAs are finite;
- perturbative theory well behaved at hard $\mu_h = m_b$ and hard-collinear $\mu_{hc} = \sqrt{\Lambda_{QCD} m_b}$ scales;

individual NNLO contributions are sizeable, while overall contributions only moderate; but may change the overall pattern of CP asymmetries;

To do next: \textit{NNLO corrections to topological penguin amplitudes}
Backup slides
Some comments on the QCD factorization approach

- **Successes**: systematic framework to calculate $\langle M_1M_2|Q_i|B\rangle$; very successful:
  - consistent global description of a variety of decay modes;
  - strong phases start at $O(\alpha_s)$, dynamical explanation of smallness of direct CP asymmetries in hadronic B decays;
  - dynamical explanation of the BR pattern of $B \rightarrow \eta^{(')}K^{(*)}$;
  - dynamical explanation of intricate pattern of penguin interference seen in $PP, PV$ and $VP$ modes;

- **Limitations**: some issues still remain to be resolved:
  - factorization of power corrections is generally broken;
  - endpoint divergences in higher-twist spectator-scattering and annihilation contributions appear, bringing large model-dependent uncertainties;
  - could not account for some data, such as large $BR(B \rightarrow \pi^0\pi^0)$, the unmatched CP asymmetries in $B \rightarrow \pi K$ decays....
Operator basis in QCD

- adopt the CMM operator basis in our calculation;
- need to introduce four extra evanescent operators;

\[ Q_1 = \bar{u} \gamma^\mu L T^A b \ [\bar{d} \gamma_\mu L T^A u] , \]
\[ Q_2 = \bar{u} \gamma^\mu L b \ [\bar{d} \gamma_\mu L u] , \]
\[ E_1 = \bar{u} \gamma^\mu \gamma^\nu \gamma^\rho L T^A b \ [\bar{d} \gamma_\mu \gamma_\nu \gamma_\rho L T^A u] - 16 Q_1 , \]
\[ E_2 = \bar{u} \gamma^\mu \gamma^\nu \gamma^\rho L b \ [\bar{d} \gamma_\mu \gamma_\nu \gamma_\rho L u] - 16 Q_2 , \]
\[ E'_1 = \bar{u} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\tau L T^A b \ [\bar{d} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\tau L T^A u] - 20 E_1 - 256 Q_1 , \]
\[ E'_2 = \bar{u} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\tau L b \ [\bar{d} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\tau L u] - 20 E_2 - 256 Q_2 , \]
SCET$_1(hc, c, s)$ operator basis

- convention in SCET: $M_1$, along $n_−, \eta_−\xi = 0$; $M_2$, along $n_+, \eta_+\chi = 0$; $h_ν$, with $\gamma h_ν = h_ν$.

- Right insertion:

\[
O_1 = \left[ \bar{\chi} \frac{\eta_−}{2}(1 - \gamma_5)\chi \right] \left[ \bar{\xi} \eta_+(1 - \gamma_5)h_ν \right], \\
O_2 = \left[ \bar{\chi} \frac{\eta_−}{2}(1 - \gamma_5)\gamma_\perp^\alpha \gamma_\perp^\beta \chi \right] \left[ \bar{\xi} \eta_+(1 - \gamma_5)\gamma_\perp^\beta \gamma_\perp^\alpha h_ν \right], \\
O_3 = \left[ \bar{\chi} \frac{\eta_−}{2}(1 - \gamma_5)\gamma_\perp^\alpha \gamma_\perp^\beta \gamma_\perp^\gamma \gamma_\perp^\delta \chi \right] \left[ \bar{\xi} \eta_+(1 - \gamma_5)\gamma_\perp^\delta \gamma_\perp^\gamma \gamma_\perp^\beta \gamma_\perp^\alpha h_ν \right].
\]

- Wrong insertion:

\[
\tilde{O}_1 = \left[ \bar{\xi} \gamma_\perp^\alpha (1 - \gamma_5)\chi \right] \left[ \bar{\chi}(1 + \gamma_5)\gamma_\perp^\alpha h_ν \right], \\
\tilde{O}_2 = \left[ \bar{\xi} \gamma_\perp^\alpha \gamma_\perp^\beta \gamma_\perp^\gamma (1 - \gamma_5)\chi \right] \left[ \bar{\chi}(1 + \gamma_5)\gamma_\perp^\alpha \gamma_\perp^\gamma \gamma_\perp^\beta h_ν \right], \\
\tilde{O}_3 = \left[ \bar{\xi} \gamma_\perp^\alpha \gamma_\perp^\beta \gamma_\perp^\gamma \gamma_\perp^\delta \gamma_\perp^\epsilon (1 - \gamma_5)\chi \right] \left[ \bar{\chi}(1 + \gamma_5)\gamma_\perp^\alpha \gamma_\perp^\epsilon \gamma_\perp^\delta \gamma_\perp^\gamma \gamma_\perp^\beta h_ν \right].
\]
1-loop Jet function calculation

- $J$: now known to NLO
  
  [Beneke,Yang 05; Becher,Hill,Lee,Neubert 04]

- just the SD coefficient from hard-collinear scale $\mu_{hc}$;

- even at the hard-collinear scale, PT is still well-behaved.
1-loop calculation for $H^{II}$

- diagrammatical approach (QCDF) \([Pilipp 07]\)
  $\iff$ the effective field theory formulation (SCET) \([Beneke,Jager 05; Kivel 06]\);

- factorization does hold at $\mu_{hc}$; PT is well-behaved;
- double lines are massive, while single lines massless;
- dots on lines denote squared propagators;
- have been cross-checked in $B \rightarrow X_u l\nu$ calculation, [Bell’07,’08; Bonciani,Ferroglia’08; Asatrian,Greub,Pecjak’08; Beneke,Huber,Li’08]
- these are extra MIs needed for hadronic B decays;
- have also been cross-checked, [Bell’07,’09; Beneke,Huber,Li’09]
Illustration of the techniques, I

- IBP IDs: for 2-loop case, 8 IDs per scalar integral

\[ \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{\partial}{\partial a^\mu} [b^\mu f(k, l, p_i)] = 0 ; \quad a^\mu = k^\mu, l^\mu ; \quad b^\mu = k^\mu, l^\mu, p_i^\mu \]

- Solve systems of these equations via Laporta algorithm
  \[ \Rightarrow \text{scalar integrals can be expressed as a linear combination of MIs:} \]

\[ = \frac{(8 - 3D)(7uD - 8D - 24u + 28)}{3(D - 4)^2 m^4_b u^3} \]
\[ - \frac{2[u^2(D - 4) + (16D - 56)(1 - u)]}{3(D - 4)^2 m^2_b u^3} \]

- Differential equations:

\[ \frac{\partial}{\partial u} \text{MI}_i(u) = f(u, \epsilon) \text{MI}_i(u) + \sum_{j \neq i} g_j(u, \epsilon) \text{MI}_j(u) \]

  - needs result from Laporta reduction.
  - boundary condition from some other techniques.

[\text{Kotikov'91; Remiddi'97}]
Illustration of the techniques, II

- **Mellin-Barnes representation:**

\[
\frac{1}{(A_1+A_2)^\alpha} = \oint_{\gamma} \frac{dz}{2\pi i} A_1^z A_2^{-\alpha-z} \frac{\Gamma(-z) \Gamma(\alpha+z)}{\Gamma(\alpha)}
\]

- partially automated,
- used as numerical cross checks of our analytic calculation,

![Diagram](image)

- Special functions frequently used:
  - HPL up to weight 4 with argument \( u \) or \( 1 - u \)
  - Polylogarithms \( \text{Li}_2, \text{Li}_3, \text{Li}_4 \) with argument \( u, 1 - u, \frac{u}{1-u} \)
  - Hypergeometric function \( pFq \), \( \epsilon \)-expansion,

  [Maitre, Huber’05, ’07]
To prove factorization, should check that $T^{I,II}$ are free of both UV- and IR-divergences, *very non-trivial both in QCD and in SCET!*

should consider the evanescent operator contributions from QCD and SCET;

some operators are non-local, needs distribution like ERBL kernel up to two-loop order,  

[**Lepage, Brodsky’80; Efremov, Radyushkin’80**]

SCET graphs contributing to the anomalous dimension $Y_{ij}$ of the SCET four-quark operators:
Dependence of $\alpha_{1,2}$ on the Gegenbauer moments.

- at LO, no such a dependence at all in QCDF/SCET;

- at NLO, the dependence is ($\mu = m_b$):
  \[
  [\alpha_1]_V = 1.040 + 0.013i - (0.007 - 0.013i) a_1^{M_2} + 0.001 a_2^{M_2}
  \]
  \[
  [\alpha_2]_V = 0.029 - 0.079i + (0.046 - 0.079i) a_1^{M_2} - 0.009 a_2^{M_2}
  \]

- at NNLO, the dependence is ($\mu = m_b$):
  \[
  [\alpha_1]_V = 1.057 + 0.038i - (0.032 - 0.022i) a_1^{M_2} + (0.003 - 0.001i) a_2^{M_2},
  \]
  \[
  [\alpha_2]_V = 0.013 - 0.126i + (0.139 - 0.096i) a_1^{M_2} - (0.021 + 0.009i) a_2^{M_2}.
  \]

- the dependence on $a_2^{M_2}$ is very small for both tree amplitudes;

- the first Gegenbauer moment $a_1^{M_2}$ is more important, especially for $[\alpha_2]_V$, furthermore enhanced at NNLO;

- a source of non-factorizable SU(3) flavour symmetry breaking.
Definition of various ratios in tree-dominated decays, I

\[ R_{\pi\pi}^{+-} \equiv 2 \frac{\Gamma(B^- \to \pi^- \pi^0)}{\Gamma(B^0 \to \pi^+ \pi^-)}, \]
\[ R_{\rho\rho}^{+-} \equiv 2 \frac{\Gamma(B^- \to \rho^- \rho^0)}{\Gamma(B^0 \to \rho^+_\rho^-)}, \]
\[ R_{\pi\rho}^{00} \equiv \frac{2 \Gamma(B^0 \to \pi^0 \rho^0)}{\Gamma(B^0 \to \pi^+ \rho^-) + \Gamma(B^0 \to \pi^- \rho^+)} . \]

- the dependence on FF and CKM factor cancels out;
- \( R^{+-} \propto |\alpha_1 + \alpha_2|^2 / |\alpha_1|^2 \), while \( R^{00} \propto |\alpha_2|^2 / |\alpha_1|^2 \)
- highlight the importance of hard spectator-scattering;
- strongly depend on the B-meson LCDA \( \lambda_B \) and \( r_{sp} = \frac{9f_{M2}\hat{f}_B}{m_b FF BM_1(0)\lambda_B} \).
Definition of various ratios in tree-dominated decays, II

\[ R_1 \equiv \frac{\Gamma(\bar{B}^0 \to \pi^+ \rho^-)}{\Gamma(\bar{B}^0 \to \pi^+\pi^-)} , \]
\[ R_3 \equiv \frac{\Gamma(\bar{B}^0 \to \pi^+ \rho^-)}{\Gamma(\bar{B}^0 \to \pi^-\rho^+)} , \]
\[ R_5 \equiv \frac{2 \Gamma(B^- \to \pi^0 \rho^-)}{\Gamma(\bar{B}^0 \to \pi^+ \rho^-)} - 1 , \]
\[ R_6 \equiv \frac{\Gamma(\bar{B}^0 \to \pi^+ \rho^-) + \Gamma(\bar{B}^0 \to \pi^- \rho^+)}{2 \Gamma(\bar{B}^0 \to \rho_L^+ \rho_L^-)} . \]

\[ R_{C\pi} = \frac{\Gamma(\bar{B}^0 \to \pi^0\pi^0)}{\Gamma(\bar{B}^0 \to \pi^0\rho^0)} , \]
\[ R_{C\rho} = \frac{\Gamma(\bar{B}^0 \to \rho_L^0\rho_L^0)}{\Gamma(\bar{B}^0 \to \rho^0\rho^0)} . \]

- \( R_1 \) is free of FF dependence; \( R_3 \propto A_0^{B\rho}(0)/f_+^{B\pi}(0) \); 
- \( R_{C\pi} \) provide information on \( \alpha_2 \) in PP, PV and VV modes; 
- \( R_{4,5,6} \) is almost free of CKM factor dependence; 
- \( R_{4,5} \) provide access to the real part of \( \alpha_2(\pi\rho) \) and \( \alpha_2(\rho\pi) \);
Various ratios of $B \to \pi\pi$, $\pi\rho$ and $\rho_L\rho_L$ decays

<table>
<thead>
<tr>
<th></th>
<th>Theory I</th>
<th>Theory II</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^{\pi\pi}_{++}$</td>
<td>$1.38_{-0.13}^{+0.12} +0.53$</td>
<td>$1.91_{-0.20}^{+0.06} +0.72$</td>
<td>$2.02 \pm 0.17$</td>
</tr>
<tr>
<td>$R^{\pi\pi}_{00}$</td>
<td>$0.09_{-0.02}^{+0.03} +0.12$</td>
<td>$0.22_{-0.05}^{+0.06} +0.28$</td>
<td>$0.60 \pm 0.08$</td>
</tr>
<tr>
<td>$R^{\rho\rho}_{++}$</td>
<td>$1.32_{-0.03}^{+0.02} +0.44$</td>
<td>$1.72_{-0.03}^{+0.01} +0.64$</td>
<td>$1.80 \pm 0.28$</td>
</tr>
<tr>
<td>$R^{\rho\rho}_{00}$</td>
<td>$0.03_{-0.00}^{+0.00} +0.07$</td>
<td>$0.10_{-0.01}^{+0.01} +0.19$</td>
<td>$0.05 \pm 0.02$</td>
</tr>
<tr>
<td>$R^{\pi\rho}_{00}$</td>
<td>$0.04_{-0.00}^{+0.00} +0.09$</td>
<td>$0.14_{-0.01}^{+0.01} +0.20$</td>
<td>$0.17 \pm 0.05$</td>
</tr>
<tr>
<td>$R_1$</td>
<td>$2.41_{-0.18}^{+0.18} +0.32$</td>
<td>$2.41_{-0.20}^{+0.00} +0.37$</td>
<td>$3.04 \pm 0.37$</td>
</tr>
<tr>
<td>$R_2$</td>
<td>$1.90_{-0.19}^{+0.13} +0.53$</td>
<td>$1.92_{-0.20}^{+0.00} +0.41$</td>
<td>$2.23 \pm 0.24$</td>
</tr>
<tr>
<td>$R_3$</td>
<td>$1.73_{-0.12}^{+0.13} +0.41$</td>
<td>$1.69_{-0.12}^{+0.00} +0.72$</td>
<td>$2.15 \pm 0.43$</td>
</tr>
<tr>
<td>$R_4$</td>
<td>$0.58_{-0.02}^{+0.02} +0.35$</td>
<td>$1.26_{-0.00}^{+0.00} +0.36$</td>
<td>$1.12_{-0.08}^{+0.00} +0.46$</td>
</tr>
<tr>
<td>$R_5$</td>
<td>$0.30_{-0.04}^{+0.05} +0.20$</td>
<td>$0.64_{-0.05}^{+0.06} +0.36$</td>
<td>$0.30_{-0.23}^{+0.00} +0.22$</td>
</tr>
<tr>
<td>$R_6$</td>
<td>$0.54_{-0.01}^{+0.01} +0.23$</td>
<td>$0.53_{-0.01}^{+0.01} +0.17$</td>
<td>$0.49 \pm 0.08$</td>
</tr>
<tr>
<td>$R^{\pi}_{C}$</td>
<td>$0.64_{-0.17}^{+0.22} +0.64$</td>
<td>$0.42_{-0.08}^{+0.09} +0.37$</td>
<td>$0.78 \pm 0.22$</td>
</tr>
<tr>
<td>$R^{\rho}_{C}$</td>
<td>$0.74_{-0.09}^{+0.10} +0.58$</td>
<td>$0.70_{-0.06}^{+0.06} +0.46$</td>
<td>$0.27_{-0.14}^{+0.10} +0.13$</td>
</tr>
</tbody>
</table>
## Input parameters used in the phenomenological analysis

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value/Range</th>
<th>Parameter</th>
<th>Value/Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda^{(5)}_{\text{MS}}$</td>
<td>$0.225$</td>
<td>$\mu_{hc}$</td>
<td>$1.5 \pm 0.6$</td>
</tr>
<tr>
<td>$m_c$</td>
<td>$1.3 \pm 0.2$</td>
<td>$f_{B_d}$</td>
<td>$0.195 \pm 0.015$</td>
</tr>
<tr>
<td>$m_s(2 \text{ GeV})$</td>
<td>$0.09 \pm 0.02$</td>
<td>$f_\pi$</td>
<td>$0.131$</td>
</tr>
<tr>
<td>$(m_u + m_d)/m_s$</td>
<td>$0.0826$</td>
<td>$f_{+}^B\pi(0)$</td>
<td>$0.25 \pm 0.05^{\dagger}$</td>
</tr>
<tr>
<td>$m_b$</td>
<td>$4.8$</td>
<td>$f_\rho$</td>
<td>$0.209$</td>
</tr>
<tr>
<td>$\bar{m}_b(\bar{m}_b)$</td>
<td>$4.2$</td>
<td>$A_0^{B\rho}(0)$</td>
<td>$0.30 \pm 0.05^{\dagger}$</td>
</tr>
<tr>
<td>$</td>
<td>V_{cb}</td>
<td>$</td>
<td>$0.0415 \pm 0.0010$</td>
</tr>
<tr>
<td>$</td>
<td>V_{ub}/V_{cb}</td>
<td>$</td>
<td>$0.09 \pm 0.02$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$(70 \pm 10)^\circ$</td>
<td>$\sigma_2(1 \text{ GeV})$</td>
<td>$3 \pm 2$</td>
</tr>
<tr>
<td>$\tau(B^-)$</td>
<td>$1.64 \text{ ps}$</td>
<td>$a_2^\pi(2 \text{ GeV})$</td>
<td>$0.2 \pm 0.15$</td>
</tr>
<tr>
<td>$\tau(B_d)$</td>
<td>$1.53 \text{ ps}$</td>
<td>$a_2^\rho(2 \text{ GeV})$</td>
<td>$0.1 \pm 0.15$</td>
</tr>
<tr>
<td>$\mu_b$</td>
<td>$4.8^{+4.8}_{-2.4}$</td>
<td>$a_2^{\rho,\perp}(2 \text{ GeV})$</td>
<td>$0.1 \pm 0.15$</td>
</tr>
</tbody>
</table>

$^{\dagger}$ $f_{+}^B\pi(0) = 0.23 \pm 0.03$, $A_0^{B\rho}(0) = 0.28 \pm 0.03$, $\lambda_B(1 \text{ GeV}) = (0.20^{+0.05}_{-0.00})$ GeV.