弦和场论圈图计算新进展

朱 传 界

中国科学院理论物理所

2006年10月31日, 桂林

主 要 内 容

1. 引言
2. Witten’s TST 和CSW理论
3. BCFW 递推关系和割线可构部分的计算
4. 有理(割线不可构)部分的计算
5. 展望

1. 引言: NLO计算的重要性

1. 实验要求更高精度的计算：LHC和ILC;
2. 场论计算的困难;
3. 场论中的方法：色分解， spinor helicity， 递推关系， SUSY关系，string-inspired 方法 (Bern-Kosower规则）
4. 最近的发展(2003年12月开始)

色 分 解

➢树图：

$$\mathcal{A}_n^{\text{tree}}(\{k_i,\lambda_i,a_i\}) = g^{n-2}\sum_{\sigma\in S_n/Z_n}\text{Tr}\,(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\;\;A_n^{\text{tree}}(\sigma(1^{\lambda_1}),\ldots,\sigma(n^{\lambda_n}))$$

➢单圈：

$$\mathcal{A}_n^{1-\text{loop}}(\{k_i,\lambda_i,a_i\}) = g^n\Bigg[\sum_{\sigma\in S_n/Z_n} N_c\,\text{Tr}\,(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\,A_{n;1}(\sigma(1^{\lambda_1}),\ldots,\sigma(n^{\lambda_n}))$$
$$+\sum_{c=2}^{\lfloor n/2\rfloor+1}\sum_{\sigma\in S_n/S_{n;c}}\text{Tr}\,(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(c-1)}})\,\text{Tr}\,(T^{a_{\sigma(c)}}\cdots T^{a_{\sigma(n)}})\,\text{A}_{n;c}(\sigma(1^{\lambda_1}),\ldots,\sigma(n^{\lambda_n}))\Bigg]$$

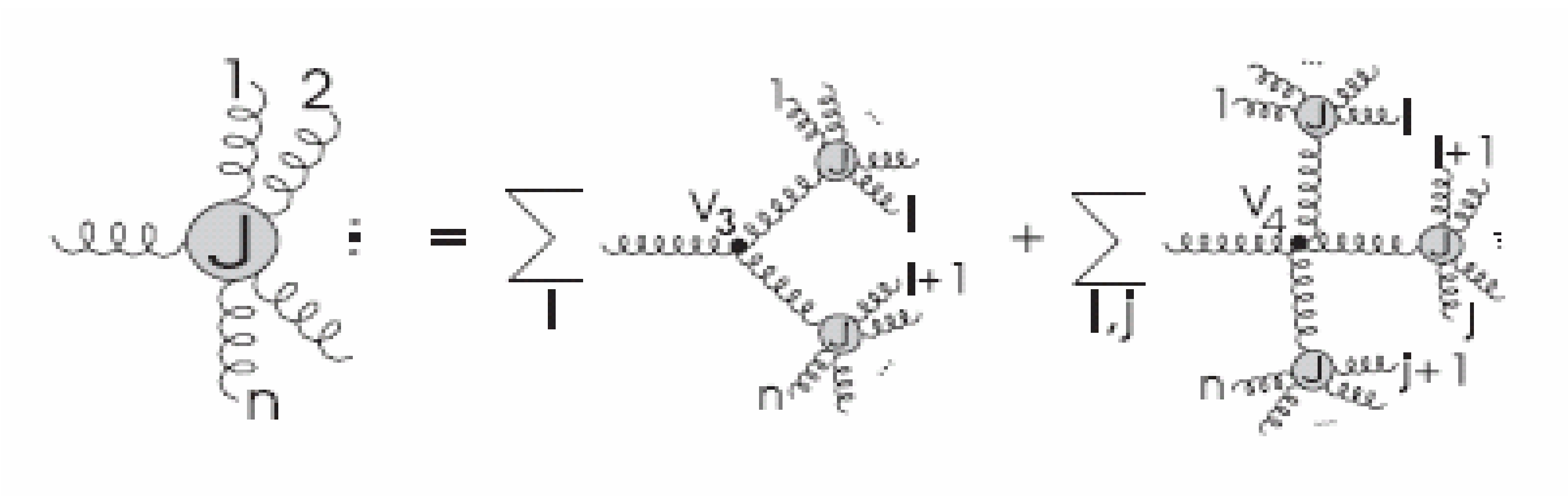
Spinor helicity (Chinese Magic)

$$\varepsilon_\mu^{(+)}(k;q) = \frac{\langle q^-|\gamma_\mu|k^-\rangle}{\sqrt{2}\langle q^-|k^+\rangle}, \qquad \varepsilon_\mu^{(-)}(k;q) = \frac{\langle q^+|\gamma_\mu|k^+\rangle}{\sqrt{2}\langle k^+|q^-\rangle},$$

$$\varepsilon_{\alpha\dot\beta}^{(+)}(k;q) = \frac{\sqrt{2}\,\eta_\alpha\tilde\lambda_{\dot\beta}}{\langle\eta\lambda\rangle}, \qquad \varepsilon_{\alpha\dot\beta}^{(-)}(k;q) = \frac{\sqrt{2}\,\lambda_\alpha\tilde\eta_{\dot\beta}}{[\lambda\eta]}.$$

$$k = \lambda\tilde\lambda \text{ and } q = \eta\tilde\eta$$

递推关系

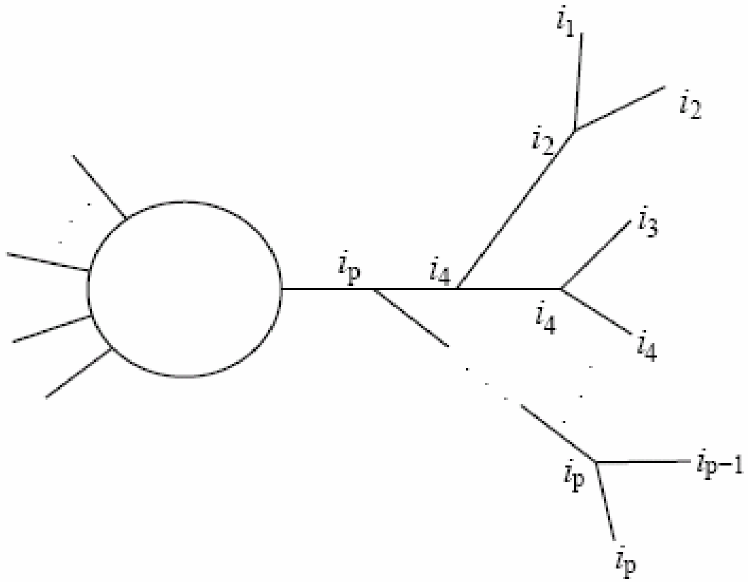


$$J^\mu(1,\ldots,n) \;=\; \frac{-i}{P_{1,n}^2}\Bigg[\sum_{i=1}^{n-1} V_3^{\mu\nu\rho}(P_{1,i},P_{i+1,n})\; J_\nu(1,\ldots,i)\; J_\rho(i+1,\ldots,n)$$
$$+\sum_{j=i+1}^{n-1}\sum_{i=1}^{n-2} V_4^{\mu\nu\rho\sigma}\;\; J_\nu(1,\ldots,i)\; J_\rho(i+1,\ldots,j)\; J_\sigma(j+1,\ldots,n)\Bigg]$$

MHV振幅:

$$A_n^{\rm tree}(1^+,\ldots,j^-,\ldots,k^-,\ldots,n^+) \;=\; i\,\frac{\langle j\,k\rangle^4}{\langle 1\,2\rangle\cdots\langle n\,1\rangle}$$

Bern-Kosower 规则



$$\mathcal{D} = \frac{(4\pi)^{\epsilon/2}}{(16\pi^2)}\Gamma(n_\ell - 2 + \epsilon/2)\int_0^1 dx_{i_{n_\ell -1}}\int_0^{x_{i_{n_\ell -1}}} dx_{i_{n_\ell -2}}\cdots\int_0^{x_{i_3}} dx_{i_2}\int_0^{x_{i_2}} dx_{i_1}$$

$$\times \frac{K_{\rm red}}{\left(\sum_{l<m}^{n_\ell} P_{i_l}\cdot P_{i_m} x_{i_m i_l}(1 - x_{i_m i_l})\right)^{n_\ell - 2 + \epsilon/2}}$$

2. Witten’s TST 和CSW理论

Twistor 空间：

- 类光矢量的旋量形式 $p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}}$ 可以写成一对旋量乘积的形式 $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$；
- 作一半的傅立叶变换:

$$\widetilde{f}(\mu) = \int \frac{d^2\widetilde{\lambda}}{(2\pi)^2} \exp(i\mu^{\dot{a}}\widetilde{\lambda}_{\dot{a}}) f(\widetilde{\lambda})$$

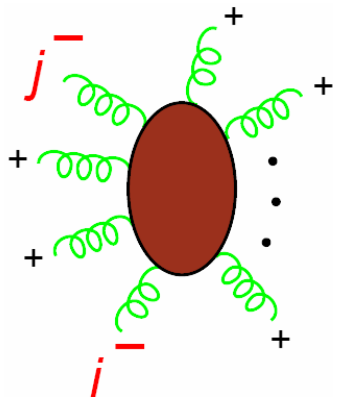
$$\tilde{\mathcal{A}}(\lambda_i, \mu_i) = \int \prod_{j=1}^{n} \frac{d^2\tilde{\lambda}_j}{(2\pi)^2} \exp(i[\mu_j, \tilde{\lambda}_j]) \mathcal{A}(\lambda_i, \tilde{\lambda}_i)$$

- Twistor 空间坐标：

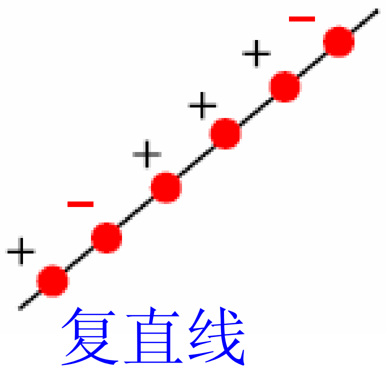
$$(\lambda_1, \lambda_2, \mu^{\dot{1}}, \mu^{\dot{2}}) \equiv (\xi\lambda_1, \xi\lambda_2, \xi\mu^{\dot{1}}, \xi\mu^{\dot{2}})$$

TST 和 Twistor 空间的振幅性质

- 一个包括p个 正helicity胶子q个负helicity胶子的*l*圈振幅,必然仅在twistor空间的一条全纯曲线上不为0, 这个曲线的阶是：d=q-1+ *l* ，亏格不大于 *l* .

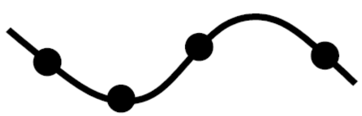


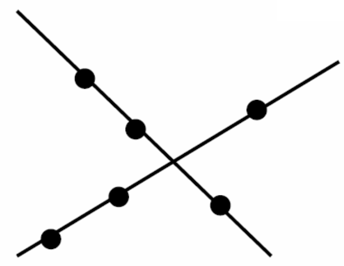
⇒



CSW 理论

$d = q - 1 + L, \quad q = \#$ negative helicities, $\quad L = \#$ loops,





- 对树图近似, 这条全纯曲线可用多条简单的直线(阶为1,亏格为0)来代替.上不为0, 这个曲线的阶是：d=q-1+ l ,亏格不大于 l .

CSW 理论计算规则

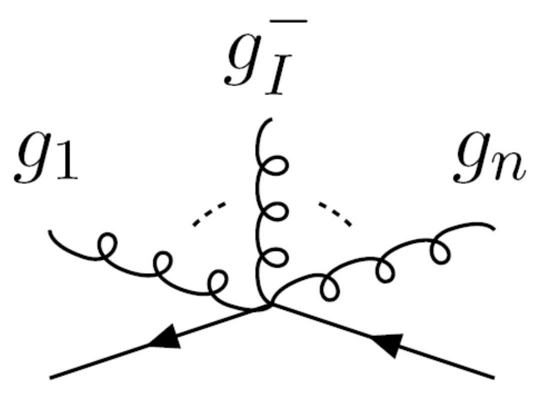
1. 将MHV顶角作off-shell continuation, 得到MHV 顶角;
2. 画出所有将MHV顶角用传播子联系起来的树图, 即MHV图;
3. 含v＋１个负helicity外线胶子的振幅的MHV图的顶角个数为v；外线分布在顶角上，要保持外线顺序。
4. 传播子 1/p^2;
5. 将符合条件的所有图加起来就得到了树图振幅。

CSW parity violated?

- CSW理论提出后,一个简单的计算证明了CSW是parity conserved.

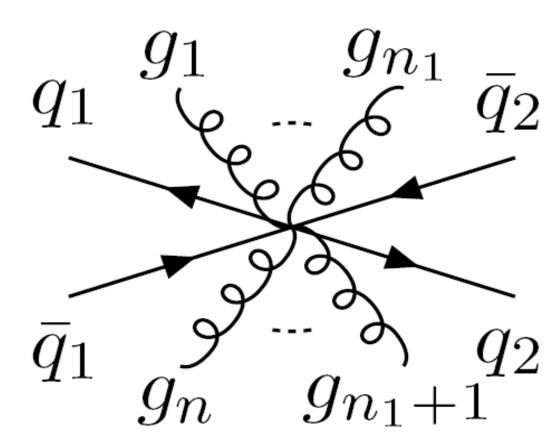
含费米子的CSW 理论

含费米子的MHV图需要增加两种费米MHV顶角



$$V(\Lambda_q^+, g_1^+, \cdots, g_I^-, \cdots, g_n^+, \Lambda_{\bar{q}}^-) = -\frac{\langle q, I\rangle\langle \bar{q}, I\rangle^3}{\langle q, 1\rangle\langle 1, 2\rangle \cdots \langle n, \bar{q}\rangle\langle \bar{q}, q\rangle}$$

$$V(\Lambda_q^-, g_1^+, \cdots, g_I^-, \cdots, g_n^+, \Lambda_{\bar{q}}^+) = \frac{\langle q, I\rangle^3\langle \bar{q}, I\rangle}{\langle q, 1\rangle\langle 1, 2\rangle \cdots \langle n, \bar{q}\rangle\langle \bar{q}, q\rangle},$$

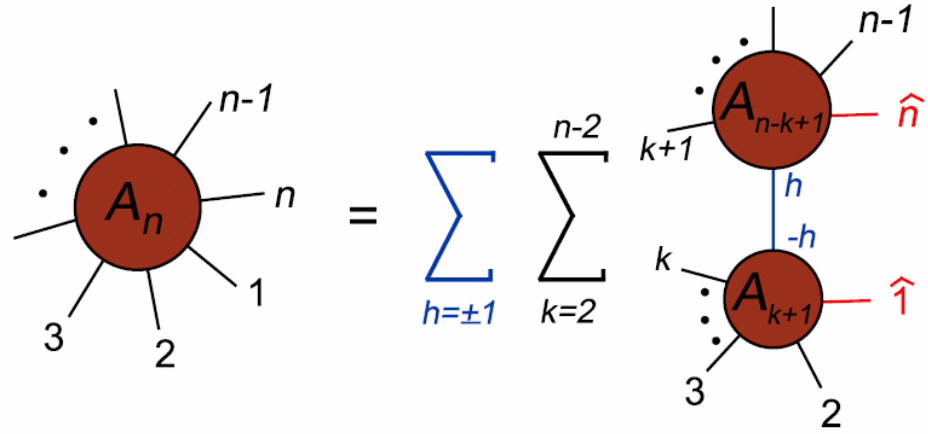


$$V(\Lambda_{q_1}^{h_1}, g_1, \cdots, g_{n_1}, \Lambda_{\bar{q}_2}^{-h_2}, \Lambda_{q_2}^{h_2}, g_{n_1+1}, \cdots, g_n, \Lambda_{\bar{q}_1}^{-h_1})$$
$$= A_0(h_1, h_2)\frac{\langle q_1, \bar{q}_2\rangle}{\langle q_1, 1\rangle\langle 1, 2\rangle \cdots \langle n_1, \bar{q}_2\rangle} \times \frac{\langle q_2, \bar{q}_1\rangle}{\langle q_2, n_1+1\rangle \cdots \langle n, \bar{q}_1\rangle}$$

3. BCFW 递推关系和割线可构部分的计算

BCFW递推关系：

$$A_n^{\text{tree}}(1,2,\ldots,n)=\sum_{h=\pm1}\sum_{k=2}^{n-2}A_{k+1}^{\text{tree}}(\hat{1},2,\ldots,k,-\hat{K}_{1,k}^{-h})\frac{i}{K^2}A_{n-k+1}^{\text{tree}}(\hat{K}_{1,k}^{h},k+1,\ldots,n-1,\hat{n})$$



- 这里 A_{k+1}^{tree} 和 A_{n-k+1}^{tree} 为低点的振幅，从而构成了递推关系;
- 对传播子helicity 求和；
- K 代表n个胶子在两个树上的分配;
- Shifted 动量仍然 massless;
- 这里的低点振幅是on shell 的物理的振幅。前面场论递推关系中低点量off –shell 且规范依赖。

$$\lambda_1 \rightarrow \hat{\lambda}_1 \equiv \lambda_1 + z_k\lambda_n, \qquad \tilde{\lambda}_1 \rightarrow \tilde{\lambda}_1,$$

$$\lambda_n \rightarrow \lambda_n, \qquad \tilde{\lambda}_n \rightarrow \hat{\tilde{\lambda}}_n \equiv \tilde{\lambda}_n - z_k\tilde{\lambda}_1$$

Status July 2006

- Status of six-gluon amplitude – progress by a lot of young people
  - analytic computation of one-loop corrections Bedford, Berger, Bern, Bidder, Bjerrum-Bohr, Brandhuber, Britto, Buchbinder, Cachazo, Dixon, Dunbar, Feng, Forde, Kosower, Mastrolia, Perkins, Spence, Travaglini, Xiao, Yang, Zhu

Amplitude	$\mathcal{N}=4$	$\mathcal{N}=1$	$\mathcal{N}=0$ cut	$\mathcal{N}=0$ rat
$--++++$	BDDK '94	BDDK '94	BDDK '94	BDK '94
$-+-+++$	BDDK '94	BDDK '94	BBST '04	BBDFK '06 XYZ '06
$-++-++$	BDDK '94	BDDK '94	BBST '04	BBDFK '06 XYZ '06
$---+++$	BDDK '94	BBDD '04	BBDI '05 BFM '06	BBDFK '06
$--+-++$	BDDK '94	BBDP '05 BBCF '05	BFM '06	XYZ '06
$-+-+-+$	BDDK '94	BBDP '05 BBCF '05	BFM '06	XYZ '06

- Numerical evaluation Ellis, Giele, Zanderighi '06

Why it is difficult?

Too many kinematic variables (8 for 5 particles with 3 mass and 9 for 6 gluons).

Gauge invariance leads to mutual cancellation.

$$I_n = \int \mathrm{d}^D p \, \frac{(\epsilon_1, p)(\epsilon_2, p - k_1) \cdots (\epsilon_n, p + k_n)}{p^2 (p - k_1)^2 \cdots (p + k_n)^2}$$

Dimension 4 quantity:

$$\frac{\textbf{Numerator: homogenous degree } m+4}{\textbf{Denominator: homogeneous degree } m}$$

Tensor reduction: very complicated intermediate expressions. For $n = 6$, $m \sim 50?$. The Gram determinant.

The Gram determinant or things as such

$$
\begin{aligned}
\Delta_6 &= -s_3^2 s_6^2 t_1^2 - t_2^2 t_3^2 t_1^2 + 2 s_3 s_6 t_2 t_3 t_1^2 \\
&+ 2 s_2 s_5 t_2 t_3^2 t_1 - 2 s_1 s_3 s_4 s_6 t_2 t_1 + 2 s_1 s_4 t_2^2 t_3 t_1 \\
&- 2 s_2 s_3 s_5 s_6 t_3 t_1 - s_1^2 s_4^2 t_2^2 - s_2^2 s_5^2 t_3^2 \\
&+ 4 s_1 s_2 s_3 s_4 s_5 s_6 - 2 s_1 s_2 s_4 s_5 t_2 t_3
\end{aligned}
$$

Something like:

$$\frac{1}{\Delta_6^{n_1} \Delta_5^{n_2} \cdots}$$

But this can be improved: $\Delta \to \sqrt{\Delta}$.

What is the rational part?

The final form of one-loop amplitude is:

$$A_g = \sum_i c_{4,i}(\epsilon, k; D)\, I_4^{D(i)}[1] + \sum_i c_{3,i}(\epsilon, k; D)\, I_3^{D(i)}[1] + \sum_i c_{2,i}(\epsilon, k; D)\, I_2^{D(i)}[1].$$

Expanding in ϵ $(D = 4 - 2\epsilon)$ gives:

$$A_g = \sum_i c_{4,i}(\epsilon, k; 4)\, I_4^{D(i)}[1] + \sum_i c_{3,i}(\epsilon, k; 4)\, I_3^{D(i)}[1] + \sum_i c_{2,i}(\epsilon, k; 4)\, I_2^{D(i)}[1] + \textbf{(Rational part)} + O(\epsilon).$$

How to compute the cut-constructible part?

割线可构部分的计算

Purpose: to determine all the rational coefficients

$$c_{n,i}(\epsilon, k; D)|_{D=4}$$

MHV and Twistor

4. 有理(割线不可构)部分的计算

- **try and error, check by numerical computation of Feynman diagrams (BDK, obsolete)**

- D**-dimensional unitarity (BDDK, Brandhuber-McNamara-Spence-Travaglini, hep-th/0506068 up to** $A_5(1^+2^+3^+4^+5^+)$**)**

- **bootstrap recursive (inspired from tree recursive relation of BCFW) (Bern-Dixon-Kosower)**

- **directly computing Feynman diagrams, keeping only term contributing to rational part. Quite old-fashion but with tricks from string theory. (Xiao-Yang-Zhu)**

Why it is feasible to compute the rational part directly from Feynman integral?

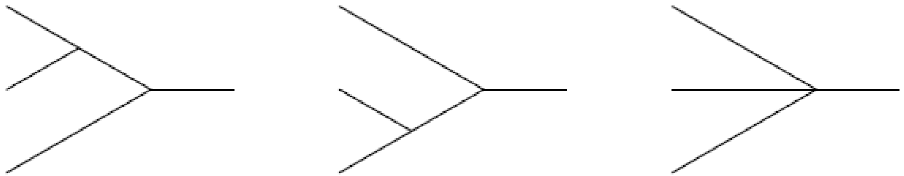
The Bern-Dunbar-Dixon-Kosower theorem:

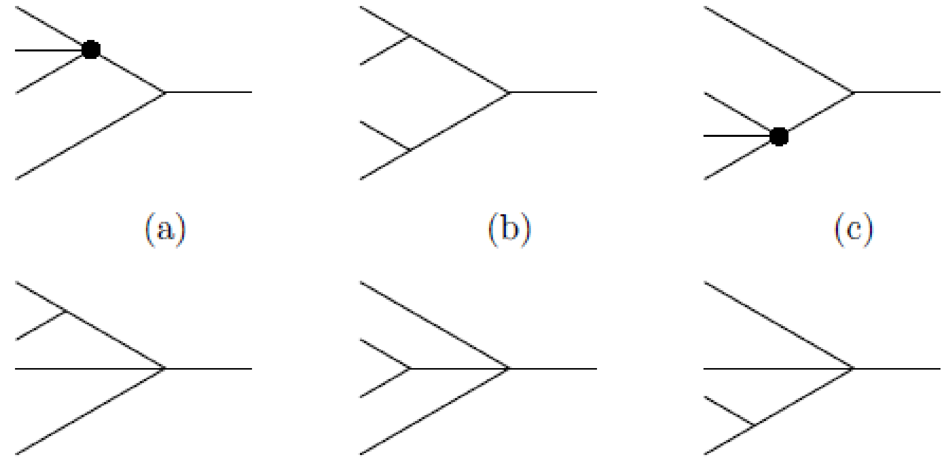
$$I_m^D[f(p)] = \int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}} \frac{f(p)}{p^2(p-K_1)^2\cdots(p+K_m)^2},$$

The rational part is 0 if $f(p)$ is a polynomial function of the internal momentum p of degree $m-2$ or less. For phenomenologically interesting models and by choosing a suitable gauge, the degree of $f(p)$ is always not greater than m.

Our strategy: do calculation while keeping only the leading and sub-leading polynomial terms in p.

Feynman diagrams and Feynman rules (I): Trees





(a) (b) (c)

(d) (e) (f)

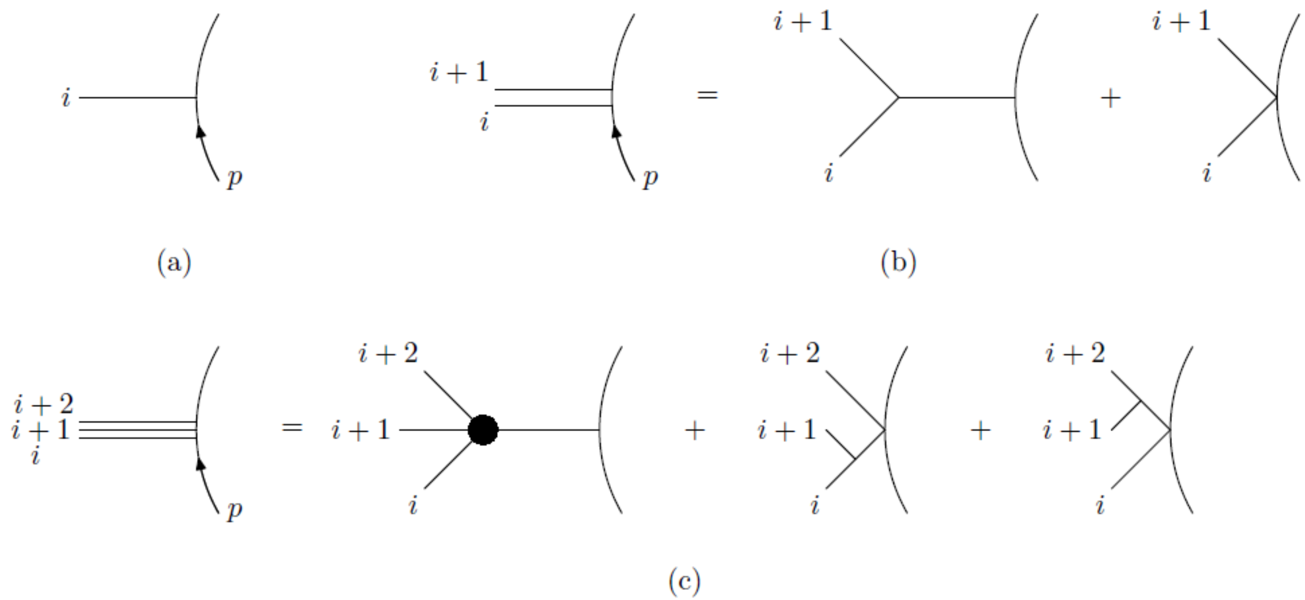
$$\epsilon_{i(i+1)} = P(\epsilon_i, k_i; \epsilon_{i+1}, k_{i+1}) \equiv \frac{1}{(k_i + k_{i+1})^2}\Big((\epsilon_i, k_{i+1})\,\epsilon_{i+1}$$

$$- (\epsilon_{i+1}, k_i)\,\epsilon_i + \frac{1}{2}(\epsilon_i, \epsilon_{i+1})\,(k_i - k_{i+1})\Big),$$

$$\epsilon_{i(i+1)(i+2)} = P(\epsilon_{i(i+1)}, k_{i(i+1)}; \epsilon_{i+2}, k_{i+2}) + P(\epsilon_i, k_i; \epsilon_{(i+1)(i+2)}, k_{(i+1)(i+2)})$$

$$+\frac{1}{s_{i(i+1)(i+2)}}\left((\epsilon_i, \epsilon_{i+2})\,\epsilon_{i+1} - \frac{1}{2}\,(\epsilon_i, \epsilon_{i+1})\,\epsilon_{i+2} - \frac{1}{2}\,(\epsilon_{i+1}, \epsilon_{i+2})\,\epsilon_i\right)$$

Feynman diagrams and Feynman rules (II): Tree to loop



(a) (b) (c)

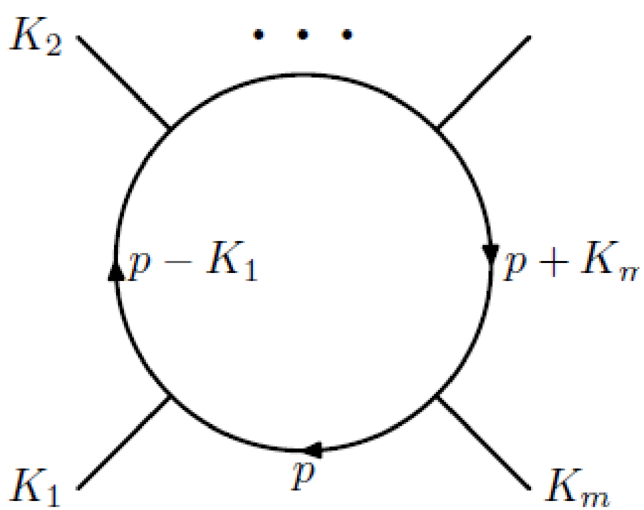
$$
\begin{aligned}
P_i(p) &= (\epsilon_i, p) = (\epsilon_i, p - k_i), \\
P_{i(i+1)}(p) &= (\epsilon_{i(i+1)}, p) - \frac{1}{2}(\epsilon_i, \epsilon_{i+1}), \\
P_{i(i+1)(i+2)}(p) &= (\epsilon_{i(i+1)(i+2)}, p) - \frac{1}{2}\left((\epsilon_{i(i+1)}, \epsilon_{i+2}) + (\epsilon_i, \epsilon_{(i+1)(i+2)})\right)
\end{aligned}
$$

Deriving the rational part (I): Feynman parametrization

By using Feynman parametrization we have

$$I_n^D[1] \equiv \int \frac{\mathrm{d}^D p}{i\pi^{D/2}} \frac{1}{p^2(p-k_1)^2 \cdots (p+k_n)^2}$$
$$= (-1)^n \, \Gamma(n-D/2) \int \mathrm{d}^n a \, \frac{\delta(1-\sum_i a_i)}{(a\cdot S\cdot a)^{n-\frac{D}{2}}},$$

where



$$a\cdot S\cdot a = \sum_{i,j=1}^{n} a_i\, a_j\, S_{ij}$$

$$S = -\frac{1}{2}\begin{pmatrix} 0 & k_1^2 & (k_1+k_2)^2 & \cdots & (k_1+k_2+\cdots k_{n-1})^2 \\ * & 0 & k_2^2 & \cdots & (k_2+k_3+\cdots k_{n-1})^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & 0 & k_{n-1}^2 \\ * & * & * & * & 0 \end{pmatrix}$$

An $n \times n$ symmetric matrix of external kinematic variables.

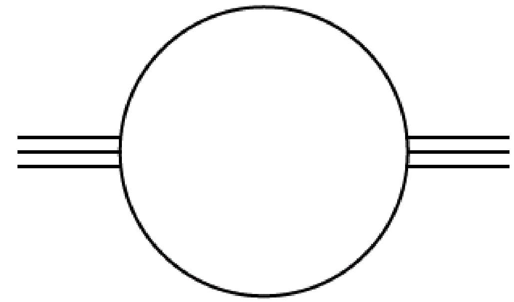
Deriving the rational part (II): recursive relations

$$
\begin{aligned}
I_n^D[g_l(a)\, a_i] &= \frac{1}{2}\,(n-1-l-D)\,\gamma_i\, I_n^{D+2}[g_l(a)] \\
&+ \frac{1}{2}\sum_j S_{ij}^{-1}\, I_{n-1}^{D(j)}[g_l(a)] + \frac{1}{2}\sum_j S_{ij}^{-1}\, I_n^{D+2}[\partial_j g_l(a)], \\
I_n^D[a_i f(a)] &= P_{ij}\,\left(I_n^{D(j)}[f(a)] + I_n^{D+2}[\partial_j f(a)]\right) + \frac{\gamma_i}{\Delta}\, I_n^D[f(a)], \\
\gamma_i &= \sum_j S_{ij}^{-1}, \qquad \Delta = \sum_i \gamma_i, \\
P_{ij} &= \frac{1}{2}\left(S_{ij}^{-1} - \frac{\gamma_i\,\gamma_j}{\Delta}\right).
\end{aligned}
$$

Deriving the rational part (III): the bubble integral

Bubble integrals:

$$I_2^D[f(p)] \equiv \int \frac{\mathrm{d}^D p}{i(\pi)^{D/2}} \frac{f(p)}{p^2 (p+K)^2}.$$



K is the sum of momenta of consecutive external gluons on one side of the bubble diagram. For $K^2 \neq 0$ we have ($D = 4 - 2\epsilon$):

$$\begin{aligned} I_2^D[1] &= \frac{\gamma_\Gamma}{\epsilon(1-2\epsilon)} (-K^2)^{-\epsilon}, \\ I_2^D[p^\mu] &= -\frac{K^\mu}{2} I_2^D[1], \\ I_2^D[a_1^2] = I_2^D[a_2^2] &= \frac{2-\epsilon}{2(3-2\epsilon)} I_2^D[1] \\ &= \frac{1}{3} I_2^D[1] + \frac{1}{18} + O(\epsilon). \end{aligned}$$

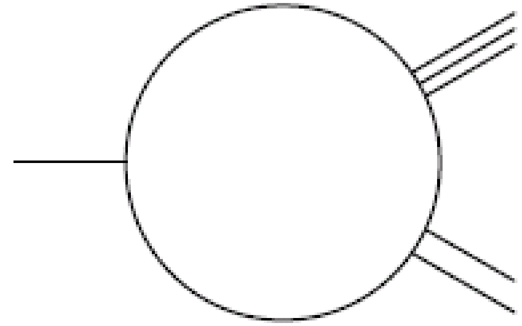
Deriving the rational part (IV): higher dim. scalar integral

$$I_n^{D+2}[1] = \frac{1}{(n-1-D)\,\Delta}\left[2\,I_n^{D}[1] - \sum_j \gamma_j\, I_{n-1}^{D(j)}[1]\right]$$

Rational part arises because I_n^D is divergent and the pre-factor depends on $D = 4 - 2\,\epsilon$.

$$\begin{aligned}
I_2^{D+2}[1] &= \frac{K^2}{9}, \qquad I_3^{D+2}[1] = \frac{1}{2}, \qquad I_4^{D+2}[1] = I_4^{D+2}[a_i] = 0, \\
I_4^{D+4}[1] &= \frac{1}{3} \times \frac{1}{2} + \frac{1}{3^2} \times 2 \times \frac{1}{2} = \frac{5}{18}, \\
I_5^{D+2}[1] &= I_5^{D+2}[a_i] = I_5^{D+2}[a_i\, a_j] = I_5^{D+4}[1] = I_5^{D+4}[a_i] = 0, \\
I_5^{D+6}[1] &= = \frac{1}{4} \times \frac{5}{18} + \frac{1}{4^2} \times 2 \times \frac{1}{6} = \frac{13}{144}.
\end{aligned}$$

Deriving the rational part (V): Triangle integral

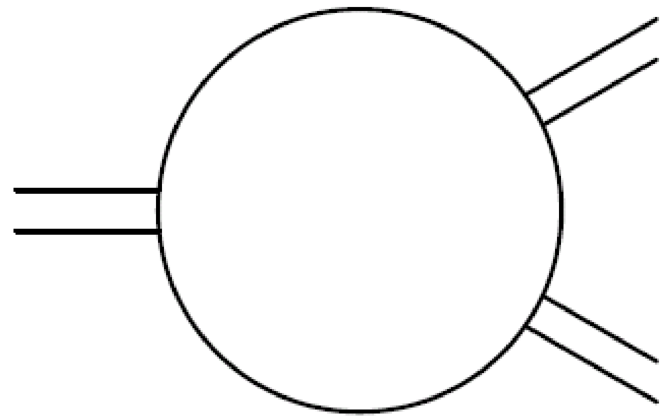


$$
\begin{aligned}
I_3^D(\epsilon_1,\epsilon_2) &\equiv -\int \frac{\mathrm{d}^D p}{i\pi^{D/2}} \frac{(\epsilon_1,p)\,(\epsilon_2,p)}{p^2(p-k_1)^2(p+K_3)^2} \\
&= \textbf{cut part} - \frac{(K_2^2+K_3^2)}{2(K_2^2-K_3^2)^2}\,(\epsilon_1,k_1)\,(\epsilon_2,k_1) - \frac{1}{2}\,(\epsilon_1,\epsilon_2) \\
&- \frac{((\epsilon_1,k_1)(\epsilon_2,k_1)+(\epsilon_1,k_1)(\epsilon_2,K_2)+(\epsilon_1,K_2)\,(\epsilon_2,k_1))}{2\,(K_2^2-K_3^2)}, \\
I_3^{D,R}(\epsilon_1,\epsilon_2) &= -\frac{(\epsilon_1,K_2)\,(\epsilon_2,k_1)}{2(K_2^2-K_3^2)} - \frac{1}{2}\,(\epsilon_1,\epsilon_2), \qquad (\epsilon_1,k_1)=0,
\end{aligned}
$$

$$
\begin{aligned}
I_3(\epsilon_i) &\equiv \int \frac{\mathrm{d}^D p}{i\pi^{D/2}} \; \frac{(\epsilon_1, p)\,(\epsilon_2, p-k_1)\,(\epsilon_3, p)}{p^2\,(p-k_1)^2\,(p+K_3)^2} \\
&= \frac{1}{36}\Big((\epsilon_2, 4\,K_2 - 7\,k_1)\,(\epsilon_1,\epsilon_3) - (2 \leftrightarrow 3) + 4(\epsilon_1, K_2)\,(\epsilon_2,\epsilon_3)\Big) \\
&- \frac{(K_2^2 + K_3^2)}{6\,(K_2^2 - K_3^2)^2}\,(\epsilon_1, K_2)\,(\epsilon_2, k_1)\,(\epsilon_3, k_1) \\
&- \frac{(\epsilon_1, K_2)\,((\epsilon_2, k_1)\,(\epsilon_3, K_3) - (\epsilon_2, K_2)\,(\epsilon_3, k_1))}{6\,(K_2^2 - K_3^2)} \\
&- \frac{(K_2^2 + K_3^2)}{12\,(K_2^2 - K_3^2)}\,((\epsilon_1,\epsilon_2)\,(\epsilon_3, k_1) + (\epsilon_1,\epsilon_3)\,(\epsilon_2, k_1))
\end{aligned}
$$

Very complicated Feynman-like rules. And indeed the 3 mass triangle is much more complicated.

$$
\begin{aligned}
I_3(\epsilon_1,\epsilon_2,\epsilon_3) \;&\equiv\; \int \frac{\mathrm{d}^D p}{i\pi^{D/2}} \frac{(\epsilon_1,p)\,(\epsilon_2,p-K_1)\,(\epsilon_3,p+K_3)}{p^2(p-K_1)^2(p+K_3)^2} \\
&= -F_0(s_1,s_2,s_3)\Big((\epsilon_1,K_1)\,(\epsilon_2,K_1)\,(\epsilon_3,K_2) \\
&\quad +(\epsilon_1,K_3)\,(\epsilon_2,K_2)\,(\epsilon_3,K_2)+(\epsilon_1,K_3)\,(\epsilon_2,K_1)\,(\epsilon_3,K_3) \\
&\quad +(\epsilon_1,K_3)\,(\epsilon_2,K_1)\,(\epsilon_3,K_2)-(\epsilon_1,K_1)\,(\epsilon_2,K_2)\,(\epsilon_3,K_3)\Big) \\
&- \sum_{i=1}^{3}(\epsilon_1,K_i)\,(\epsilon_2,K_i)\,(\epsilon_3,K_i)\,F_i(s_1,s_2,s_3) \\
&- \frac{1}{2\,\Delta}\Big((s_1-s_2-s_3)\,(\epsilon_1,K_1)\,(\epsilon_2,K_1)\,(\epsilon_3,K_3) \\
&\quad +(s_2-s_3-s_1)\,(\epsilon_1,K_1)\,(\epsilon_2,K_2)\,(\epsilon_3,K_2) \\
&\quad +(s_3-s_1-s_2)\,(\epsilon_1,K_3)\,(\epsilon_2,K_2)\,(\epsilon_3,K_3)\Big) \\
&+ \frac{7}{36}\Big((\epsilon_1,\epsilon_2)\,(\epsilon_3,K_3-K_2)+(\epsilon_2,\epsilon_3)\,(\epsilon_1,K_1-K_3) \\
&\quad +(\epsilon_3,\epsilon_1)\,(\epsilon_2,K_2-K_1)\Big) \\
&+ \frac{1}{12\,\Delta}\Big((\epsilon_1,\epsilon_2)\,(\epsilon_3,K_3-K_2)\,s_1(s_2+s_3-s_1) \\
&\quad +(\epsilon_2,\epsilon_3)\,(\epsilon_1,K_1-K_3)\,s_2(s_3+s_1-s_2) \\
&\quad +(\epsilon_3,\epsilon_1)\,(\epsilon_2,K_2-K_1)\,s_3(s_1+s_2-s_3)\Big) \\
&+ \frac{1}{12\,\Delta}\Big((\epsilon_1,\epsilon_2)\,(\epsilon_3,K_1)\,(s_3-s_2)(s_2+s_3-s_1) \\
&\quad +(\epsilon_2,\epsilon_3)\,(\epsilon_1,K_2)\,(s_1-s_3)(s_3+s_1-s_2) \\
&\quad +(\epsilon_3,\epsilon_1)\,(\epsilon_2,K_3)\,(s_2-s_1)(s_1+s_2-s_3)\Big),
\end{aligned}
$$



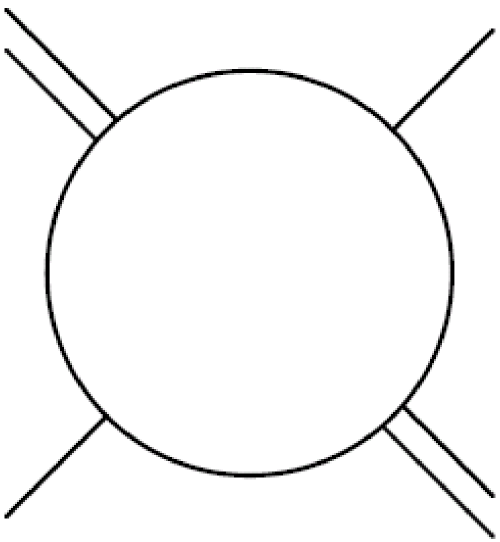
$$
\begin{aligned}
F_0(s_1,s_2,s_3) &= \frac{10\,s_1s_2s_3}{3\Delta^2}+\frac{(s_1+s_2+s_3)}{6\Delta}, \\
F_1(s_1,s_2,s_3) &= \frac{5\,(s_1+s_2-s_3)\,s_2s_3}{3\Delta^2}+\frac{(s_1-s_3)}{3\Delta}, \\
F_2(s_1,s_2,s_3) &= \frac{5\,(s_2+s_3-s_1)\,s_3s_1}{3\Delta^2}+\frac{(s_2-s_1)}{3\Delta}, \\
F_3(s_1,s_2,s_3) &= \frac{5\,(s_3+s_1-s_2)\,s_1s_2}{3\Delta^2}+\frac{(s_3-s_2)}{3\Delta}, \\
\Delta &= s_1^2+s_2^2+s_3^2-2\,(s_1\,s_2+s_2\,s_3+s_3\,s_1)
\end{aligned}
$$

Deriving the rational part (VI): 2 mass easy box integral

The basic strategy: decomposition into simple ones (equivalent to tensor reduction)

For 2 mass easy box with $\epsilon_1 = \eta_1 \tilde{\lambda}_1$ and $\epsilon_3 = \eta_3 \tilde{\lambda}_3$, we have:

$$
\begin{aligned}
I_4^{D,R}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) &= -\frac{\langle \eta_1\, 1\rangle \langle \eta_3\, 3\rangle}{\langle 1\, 3\rangle^2} I_4^{D,R}(\lambda_3 \tilde{\lambda}_1, \epsilon_2, \lambda_1 \tilde{\lambda}_3, \epsilon_4) \\
&+ \frac{\langle \eta_1\, 3\rangle \langle \eta_3\, 1\rangle}{\langle 1\, 3\rangle^2} I_4^{D,R}(k_1, \epsilon_2, k_3, \epsilon_4,) \\
+\frac{\langle \eta_1\, 3\rangle}{\langle 1\, 3\rangle} I_4^{D,R}(k_1, \epsilon_2, \epsilon_3, \epsilon_4) &+ \frac{\langle \eta_3\, 1\rangle}{\langle 3\, 1\rangle} I_4^{D,R}(\epsilon_1, \epsilon_2, k_3, \epsilon_4).
\end{aligned}
$$



$$
\begin{aligned}
I_4^{D,R}(\epsilon_1,\epsilon_2,k_3,\epsilon_4) = & \frac{K_2^2+s}{6(K_2^2-s)^2}\,(\epsilon_1,K_2)(\epsilon_2,k_1)(\epsilon_4,k_1) \\
+ & \frac{K_4^2+t}{6(K_4^2-t)^2}\,(\epsilon_1,K_4)(\epsilon_2,k_1)(\epsilon_4,k_1) \\
+ & \frac{1}{12}\left[\frac{K_2^2+s}{K_2^2-s}+\frac{K_4^2+t}{K_4^2-t}\right]\,((\epsilon_1,\epsilon_2)(\epsilon_4,k_1)+(\epsilon_1,\epsilon_4)(\epsilon_2,k_1)) \\
+ & \frac{(\epsilon_1,K_2)}{6(K_2^2-s)}\,(\epsilon_2,k_1)(\epsilon_4,k_3)+\frac{(\epsilon_1,K_4)}{6(K_4^2-t)}\,(\epsilon_2,k_3)(\epsilon_4,k_1) \\
+ & \frac{1}{9}((\epsilon_1,\epsilon_2)\,\epsilon_4+(\epsilon_1,\epsilon_4)\,\epsilon_2+(\epsilon_2,\epsilon_4)\,\epsilon_1,k_3)
\end{aligned}
$$

where $s=(k_1+K_2)^2$ and $t=(K_2+k_3)^2$. Invariant under the interchange $2\leftrightarrow 4$ $(s\leftrightarrow t)$.

By setting $\epsilon_1 = k_1$ we get

$$I_4^{D,R}(k_1, \epsilon_2, k_3, \epsilon_4)$$
$$= \frac{1}{18}(2(k_1, k_3)(\epsilon_2, \epsilon_4) - ((\epsilon_2, k_1)(\epsilon_4, k_3) + (\epsilon_2, k_3)(\epsilon_4, k_1))).$$

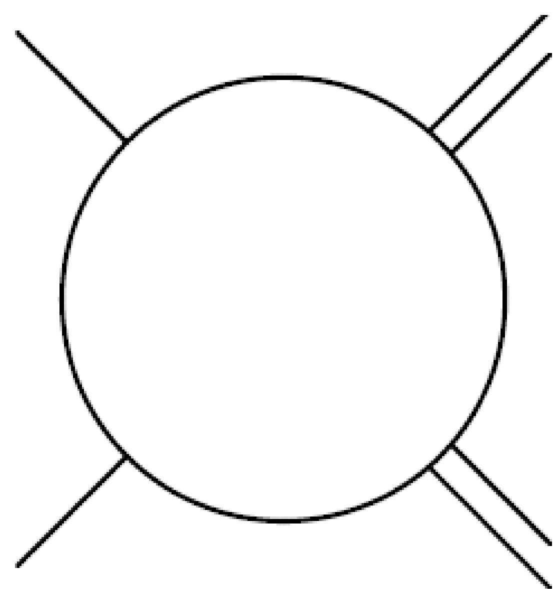
One more:

$$I_4^{D,R}(\lambda_3\tilde{\lambda}_1, \epsilon_2, \lambda_1\tilde{\lambda}_3, \epsilon_4) = \frac{4}{9}\Big((\epsilon_2, k_1)(\epsilon_4, k_3) + (\epsilon_2, k_3)(\epsilon_4, k_1)\Big)$$
$$- \frac{5}{9}(k_1, k_3)(\epsilon_2, \epsilon_4) - \frac{1}{4}\left(\frac{K_2^2 + s}{K_2^2 - s} + \frac{K_4^2 + t}{K_4^2 - t}\right)(\epsilon_2, k_1)(\epsilon_4, k_1)$$
$$- \frac{1}{4}\left(\frac{K_2^2 + t}{K_2^2 - t} + \frac{K_4^2 + s}{K_4^2 - s}\right)(\epsilon_2, k_3)(\epsilon_4, k_3).$$

Deriving the rational part (VII): 2 mass hard box integral

$$
\begin{aligned}
&I_4^{2mh}(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4;c_3,c_4)\\
&\quad\equiv I_4[(\epsilon_1,p)(\epsilon_2,p-k_1)((\epsilon_3,p+K_4)+c_3)((\epsilon_4,p+K_4)+c_4)]\\
&\quad=\int\frac{\mathrm{d}^Dp}{i\pi^{D/2}}\frac{(\epsilon_1,p)(\epsilon_2,p-k_1)((\epsilon_3,p+K_4)+c_3)((\epsilon_4,p+K_4)+c_4)}{p^2\,(p-k_1)^2\,(p-k_{12})^2\,(p+K_4)^2},
\end{aligned}
$$

$$
\begin{aligned}
&I_4^{2mh}(\lambda_1\tilde\eta_1,\eta_2\tilde\lambda_2,\epsilon_3,\epsilon_4;c_3,c_4)\\
&= \frac{1}{\langle 2|K_3|1]}\Big[-\frac{1}{6}\langle\eta_2|k_2K_3k_1|\tilde\eta_1\rangle\,(\epsilon_3,\epsilon_4)-t\,\langle\eta_2\,2\rangle\,[\tilde\eta_1\,1]\,I_4^{2mh}(\lambda_1\tilde\lambda_2,\epsilon_3,\epsilon_4)\\
&+ t\,\langle\eta_2|k_2|\tilde\eta_1]\,I_3^{3m}(\epsilon_3,\epsilon_4)+\langle\eta_2\,2\rangle\,[\tilde\eta_1\,1]\,\Big(\frac{1}{2}(\langle 1|\epsilon_3|2]\,c_4+\langle 1|\epsilon_4|2]\,c_3)\\
&\quad+\frac{1}{18}(\langle 1|\epsilon_3|2]\,\epsilon_4+\langle 1|\epsilon_4|2]\,\epsilon_3,7\,k_1+2\,k_2+9\,K_4)\Big)\\
&+ \frac{1}{18}\langle\eta_2\,2\rangle\,[\tilde\eta_1\,2]\,((\epsilon_3,k_{12})\,(\epsilon_4,k_{12})-2\,s_{12}\,(\epsilon_3,\epsilon_4))\\
&+ \frac{1}{18}\langle\eta_2|(k_2+K_3)|\tilde\eta_1]((\epsilon_3,k_2+K_3)(\epsilon_4,k_2+K_3)-2\,t\,(\epsilon_3,\epsilon_4))\\
&- \frac{1}{18}\langle\eta_2|K_3|\tilde\eta_1]\,((\epsilon_3,K_3)(\epsilon_4,K_3)-2K_3^2(\epsilon_3,\epsilon_4))\\
&+ \langle 2|K_3|\tilde\eta_1\rangle(I_3^{2m}(\eta_2\tilde\lambda_2,\epsilon_3,\epsilon_4)\\
&+ I_3^{2m}(\eta_2\tilde\lambda_2,(c_3-(\epsilon_3,K_3))\epsilon_4+(c_4+(\epsilon_4,K_4+k_1))\epsilon_3))\\
&+ I_3^{3m}(v,\epsilon_3,\epsilon_4)+I_3^{3m}(v,(c_3-(\epsilon_3,K_3))\epsilon_4+c_4\epsilon_3),\\
&+ \langle\eta_2|K_4|1]\,(\tilde I_3^{2m}(\lambda_1\tilde\eta_1,\epsilon_3,\epsilon_4)\\
&+ \tilde I_3^{2m}(\lambda_1\tilde\eta_1,(c_3-(\epsilon_3,k_2+K_3))\epsilon_4+(c_4+(\epsilon_4,K_4))\epsilon_3)\Big],
\end{aligned}
$$



where

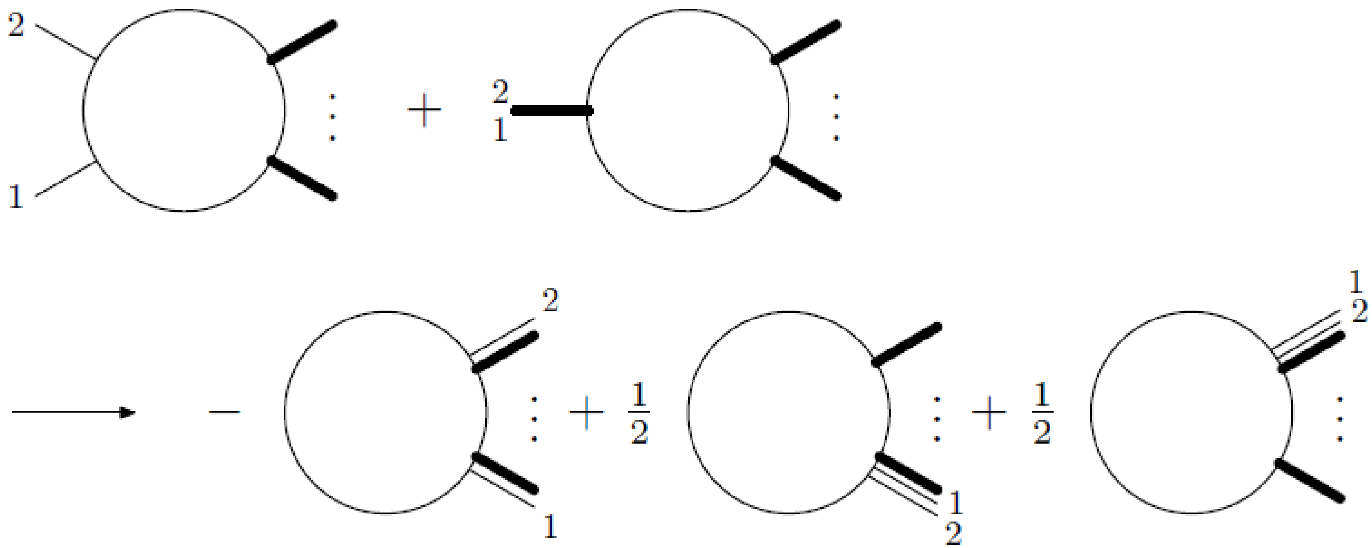
$$
v=\langle\eta_2|K_3|1]\lambda_1\tilde\eta_1+\langle\eta_2|K_3|2]\lambda_2\tilde\eta_1-(k_2,K_3)\eta_2\tilde\eta_1.
$$

The rational part (VIII): Pentagon, hexagon and higher point

Basic method: tensor reduction

For 2^+3^+, $\epsilon_2 = \lambda_3\tilde{\lambda}_2$ **and** $\epsilon_3 = \lambda_2\tilde{\lambda}_3$, **we have**

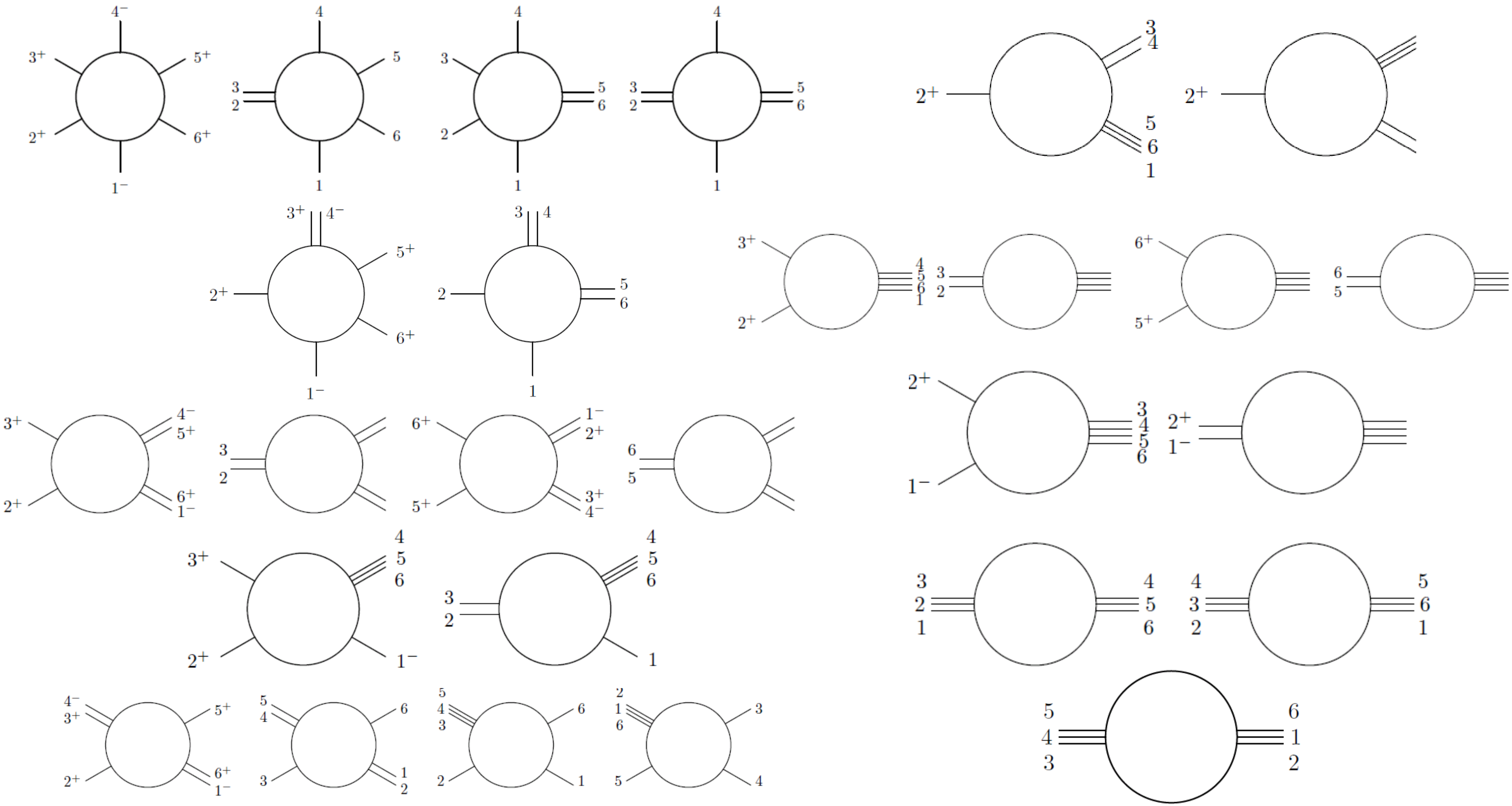
$$\frac{(\epsilon_2, p)(\epsilon_3, p - k_2)}{p^2(p-k_2)^2(p-k_{23})^2)} + \frac{(\epsilon_{23}, p) - (\epsilon_2, \epsilon_3)/2}{p^2(p-k_{23})^2}$$
$$= \quad -\frac{1}{(p-k_2)^2} + \frac{1/2}{p^2} + \frac{1/2}{(p-k_{23})^2}.$$



For $2^+3^-4^+$, **much more complicated.**

MHV: $A_6(1^-2^+3^+4^-5^+6^-)$

Total of 51 diagrams. Can be classified into the following 10 sets. Computed them in sets.



The results for 7 sets (31 diagrams) are:

$$R_{10}(1) = -\frac{1}{36}\, s_{345}(\epsilon_{345}, \epsilon_{612}), \qquad R_9(1) = -\frac{1}{18}\, s_{123}(\epsilon_{123}, \epsilon_{456}),$$

$$R_8(1) = -\frac{1}{9}\, s_{12}\, (\epsilon_{12}, \epsilon_{3456}),$$

$$R_7(1) = \frac{1}{8}\, s_{23}\, ((\epsilon_{45}, \epsilon_{61}) + (\epsilon_4, \epsilon_{561}) + (\epsilon_{456}, \epsilon_1)) - \frac{1}{12}\, s_{23}(\epsilon_{4561}, k_2),$$

$$R_1(1) = -\frac{1}{36}\left((\epsilon_1, k_2)\,(\epsilon_4, k_{12}) + \frac{1}{2}(\epsilon_1, k_{612})\,(\epsilon_4, k_{345}) + \frac{1}{4}(\epsilon_1, k_{123})\,(\epsilon_4, k_{456})\right) + \frac{1}{72}(4s_{12} - 2s_{345} - s_{123})\,(\epsilon_1, \epsilon_4),$$

$$R_3(1) = \frac{1}{18}\,(\epsilon_1, k_2)\,(\epsilon_{456}, k_3) + \frac{1}{18}\,(2s_{12} - s_{123} - 3s_{23})\,(\epsilon_1, \epsilon_{456}),$$

$$R_4(1) = \frac{1}{36}\,(\epsilon_{45}, k_3)\,(\epsilon_{61}, k_2) + \frac{1}{36}\,(2s_{345} - s_{45} - s_{61} - 3s_{23})\,(\epsilon_{45}, \epsilon_{61}).$$

The pure rational part is

$$
\begin{aligned}
R_0 \;=\; & -\frac{1}{36}\, s_{345}(\epsilon_{345},\epsilon_{612}) - \frac{1}{18}\, s_{123}(\epsilon_{123},\epsilon_{456}) - \frac{1}{9}\, s_{12}\,(\epsilon_{12},\epsilon_{3456}) \\
& + \frac{1}{8}\, s_{23}\,((\epsilon_{45},\epsilon_{61}) + (\epsilon_4,\epsilon_{561}) + (\epsilon_{456},\epsilon_1)) - \frac{1}{12}\, s_{23}(\epsilon_{4561},k_2) \\
& - \frac{1}{36}\Big((\epsilon_1,k_2)\,(\epsilon_4,k_{12}) + \frac{1}{2}(\epsilon_1,k_{612})\,(\epsilon_4,k_{345}) \\
& + \frac{1}{4}(\epsilon_1,k_{123})\,(\epsilon_4,k_{456})\Big) + \frac{1}{72}(4s_{12} - 2s_{345} - s_{123})\,(\epsilon_1,\epsilon_4) \\
& + \frac{1}{18}\,(\epsilon_1,k_2)\,(\epsilon_{456},k_3) + \frac{1}{18}\,(2s_{12} - s_{123} - 3s_{23})\,(\epsilon_1,\epsilon_{456}) \\
& + \frac{1}{36}\,(\epsilon_{45},k_3)\,(\epsilon_{61},k_2) + \frac{1}{36}\,(2s_{345} - s_{45} - s_{61} - 3s_{23})\,(\epsilon_{45},\epsilon_{61}) \\
& - \frac{1}{18}((\epsilon_2,\epsilon_{34})\epsilon_1 + (\epsilon_2,\epsilon_1)\epsilon_{34} + (\epsilon_1,\epsilon_{34})\epsilon_2, k_5 - k_6) \\
& - \frac{1}{24}\,((\epsilon_2,\epsilon_{34})(\epsilon_1,k_2) + (\epsilon_2,\epsilon_1)(\epsilon_{34},k_2)) + \frac{1}{12s_{12}}(\epsilon_2,k_1)(\epsilon_{34},k_1 + 2k_2)(\epsilon_1,k_2) \\
& + \frac{1}{8s_{12}}\,(\epsilon_3,\epsilon_4)(\epsilon_1,k_2)(\epsilon_2,k_1) + \frac{\langle 2\,5\rangle}{4\langle 2\,6\rangle}\frac{(\epsilon_2,k_1)}{s_{12}}(\epsilon_1,k_2)((\epsilon_3,\epsilon_{45}) + (\epsilon_{34},\epsilon_5)) \\
& + \frac{\langle 2\,6\rangle}{9\langle 2\,5\rangle}((\epsilon_2,\epsilon_{34})\epsilon_{61} + (\epsilon_2,\epsilon_{61})\epsilon_{34} + (\epsilon_{34},\epsilon_{61})\epsilon_2, k_5) \\
& + \frac{\langle 2\,5\rangle}{\langle 2\,6\rangle}\Big[\frac{1}{9}\,((\epsilon_2,\epsilon_{345})\epsilon_1 + (\epsilon_2,\epsilon_1)\epsilon_{345} + (\epsilon_{345},\epsilon_1)\epsilon_2, k_6) \\
& - \frac{1}{12}\,((\epsilon_2,\epsilon_{345})\epsilon_1 + (\epsilon_2,\epsilon_1)\epsilon_{345}, k_2) + \frac{1}{6\,s_{12}}\,(\epsilon_1,k_2)(\epsilon_2,k_1)(\epsilon_{345},k_2 - k_6)\Big] \\
& + (k_2,k_5)\,(\epsilon_{34},\epsilon_{61}\,\Big[\frac{5}{18}\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,5\rangle^2} - \frac{1}{18}\frac{\langle 3\,5\rangle\langle 2\,6\rangle}{\langle 2\,5\rangle^2}\Big] \\
& + ((\epsilon_{34},k_2)(\epsilon_{61},k_5) + (\epsilon_{34},k_5)(\epsilon_{61},k_2))\,\Big[-\frac{2}{9}\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,5\rangle^2} + \frac{1}{36}\frac{\langle 3\,5\rangle\langle 2\,6\rangle}{\langle 2\,5\rangle^2}\Big] \\
& + (k_2,k_6)(\epsilon_1,\epsilon_{345})\,\Big[-\frac{5}{18}\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,6\rangle^2} - \frac{1}{18}\frac{\langle 3\,6\rangle\langle 2\,5\rangle}{\langle 2\,6\rangle^2}\Big] \\
& + (\epsilon_1,k_2)(\epsilon_{345},k_6)\,\Big[\frac{4}{9}\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,6\rangle^2} + \frac{1}{18}\frac{\langle 3\,6\rangle\langle 2\,5\rangle}{\langle 2\,6\rangle^2}\Big] \\
& + \frac{1}{36}\,(7(\epsilon_2,\epsilon_{34})(\epsilon_{561},k_2) - 7(\epsilon_2,\epsilon_{561})(\epsilon_{34},k_2) + 4(\epsilon_{34},\epsilon_{561})(\epsilon_2,k_{34})) \\
& - \frac{1}{4}(\epsilon_3,\epsilon_4)\,(\epsilon_2,\epsilon_{561}) - \frac{1}{4}((\epsilon_5,\epsilon_{61}) + (\epsilon_{56},\epsilon_1))\,(\epsilon_2,\epsilon_{34}) \\
& + \frac{1}{36}\,(7(\epsilon_2,\epsilon_{345})(\epsilon_{61},k_2) - 7(\epsilon_2,\epsilon_{61})(\epsilon_{345},k_2) - 4(\epsilon_{345},\epsilon_{61})(\epsilon_2,k_{61})) \\
& - \frac{1}{4}(\epsilon_6,\epsilon_1)(\epsilon_2,\epsilon_{345}) - \frac{1}{4}((\epsilon_3,\epsilon_{45}) + (\epsilon_{34},\epsilon_5))(\epsilon_2,\epsilon_{61}) \\
& + \frac{1}{18}\,(2(\epsilon_2,\epsilon_{3456})(\epsilon_1,k_2) - 5(\epsilon_2,\epsilon_1)(\epsilon_{3456},k_2) - 2(\epsilon_{3456},\epsilon_1)(\epsilon_2,k_1)) \\
& - \frac{1}{4}((\epsilon_3,\epsilon_{456}) + (\epsilon_{345},\epsilon_6) + (\epsilon_{34},\epsilon_{56}))\,\Big[(\epsilon_2,\epsilon_1) - \frac{(\epsilon_2,k_1)(\epsilon_1,k_2)}{s_{12}}\Big] \\
& + \frac{1}{6\,s_{12}}\,(\epsilon_1,k_2)(\epsilon_2,k_1)(\epsilon_{3456},k_2) + (\epsilon_1,k_2)(\epsilon_{345},k_2)\,\frac{1}{4}\,\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,6\rangle^2}.
\end{aligned}
$$

The simple pole terms are:

$$
\begin{aligned}
R_1 \;=\; & \frac{(\epsilon_2,k_{61})}{6(s_{61} - s_{345})}\,((\epsilon_{34},k_2)(\epsilon_1,k_6) - (\epsilon_{34},k_5)(\epsilon_1,k_2)) \\
& - \frac{(\epsilon_2,k_{34})}{12(s_{34} - s_{234})}\,(\epsilon_{34},k_2)(\epsilon_1,k_{56}) \\
& - \frac{s_{61} + s_{345}}{12(s_{61} - s_{345})}\,((\epsilon_2,\epsilon_{34})(\epsilon_1,k_2) + (\epsilon_2,\epsilon_1)(\epsilon_{34},k_2)) \\
& - \frac{s_{34} + s_{234}}{24(s_{34} - s_{234})}\,((\epsilon_2,\epsilon_{34})(\epsilon_1,k_2) + (\epsilon_2,\epsilon_1)(\epsilon_{34},k_2)) \\
& + \frac{(\epsilon_3,\epsilon_4)(\epsilon_1,k_2)}{8}\,\Big[\frac{2(\epsilon_2,k_{61})}{s_{61} - s_{345}} - \frac{(\epsilon_2,k_{34})}{s_{34} - s_{234}}\Big] \\
& + \frac{\langle 2\,6\rangle}{4\langle 2\,5\rangle}\,\Big[\frac{(\epsilon_2,k_{34})}{s_{34} - s_{234}} - \frac{(\epsilon_2,k_{61})}{s_{61} - s_{345}}\Big]\,(\epsilon_3,\epsilon_4)(\epsilon_{61},k_2) \\
& + \frac{\langle 2\,6\rangle}{4\langle 2\,5\rangle}\,\Big[\frac{(\epsilon_2,k_{34})}{s_{34} - s_{234}} - \frac{(\epsilon_2,k_{61})}{s_{61} - s_{345}}\Big]\,(\epsilon_1,\epsilon_6)(\epsilon_{34},k_2) \\
& + \frac{\langle 2\,5\rangle}{4\langle 2\,6\rangle}\,\Big[\frac{(\epsilon_2,k_{61})}{s_{61} - s_{345}}\Big]\,(\epsilon_1,k_2)((\epsilon_3,\epsilon_{45}) + (\epsilon_{34},\epsilon_5)) \\
& + \frac{\langle 2\,6\rangle}{\langle 2\,5\rangle}\,\Big[\frac{1}{12}\,\Big[\frac{s_{34} + s_{234}}{s_{34} - s_{234}} + \frac{s_{61} + s_{345}}{s_{61} - s_{345}}\Big]\,((\epsilon_2,\epsilon_{34})\epsilon_{61} + (\epsilon_2,\epsilon_{61})\epsilon_{34}, k_2) \\
& + \Big[\frac{(\epsilon_2,k_{34})(\epsilon_{34},k_2)(\epsilon_{61},k_5)}{6(s_{34} - s_{234})} + \frac{(\epsilon_2,k_{61})(\epsilon_{34},k_5)(\epsilon_{61},k_2)}{6(s_{61} - s_{345})}\Big]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\langle 2\,5\rangle}{\langle 2\,6\rangle}\left[\frac{s_{61}+s_{345}}{12(s_{61}-s_{345})}((\epsilon_2,\epsilon_{345})\epsilon_1+(\epsilon_2,\epsilon_1)\epsilon_{345},k_2)\right.\\
&+ \left.(\epsilon_2,k_{61})(\epsilon_{345},k_2)\frac{(\epsilon_1,k_6)}{6(s_{61}-s_{345})}\right]\\
&+ \frac{s_{61}+s_{345}}{s_{61}-s_{345}}\left[\frac{1}{4}\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,5\rangle^2}(\epsilon_{34},k_2)(\epsilon_{61},k_2)+\frac{1}{4}\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,6\rangle^2}(\epsilon_{345},k_2)(\epsilon_1,k_2)\right]\\
&+ \frac{s_{34}+s_{234}}{s_{34}-s_{234}}\frac{1}{4}\frac{\langle 2\,3\rangle\langle 5\,6\rangle}{\langle 2\,5\rangle^2}(\epsilon_{34},k_2)(\epsilon_{61},k_2)\\
&- \frac{s_{34}+s_{234}}{12(s_{34}-s_{234})}((\epsilon_2,\epsilon_{34})(\epsilon_{561},k_2)+(\epsilon_2,\epsilon_{561})(\epsilon_{34},k_2))\\
&- \frac{1}{4}(\epsilon_3,\epsilon_4)\frac{(\epsilon_2,k_{34})(\epsilon_{561},k_2)}{s_{34}-s_{234}}-\frac{1}{4}((\epsilon_5,\epsilon_{61})+(\epsilon_{56},\epsilon_1))\frac{(\epsilon_2,k_{34})(\epsilon_{34},k_2)}{s_{34}-s_{234}}\\
&+ \frac{s_{61}+s_{345}}{12(s_{61}-s_{345})}((\epsilon_2,\epsilon_{345})(\epsilon_{61},k_2)+(\epsilon_2,\epsilon_{61})(\epsilon_{345},k_2))\\
&- \frac{1}{4}(\epsilon_6,\epsilon_1)\frac{(\epsilon_2,k_{61})(\epsilon_{345},k_2)}{s_{61}-s_{345}}-\frac{1}{4}((\epsilon_3,\epsilon_{45})+(\epsilon_{34},\epsilon_5))\frac{(\epsilon_2,k_{61})(\epsilon_{61},k_2)}{s_{61}-s_{345}}.
\end{aligned}
$$

The double pole terms are:

$$
\begin{aligned}
R_2 &= \frac{s_{61}+s_{345}}{6(s_{61}-s_{345})^2}\left[(\epsilon_2,k_{61})(\epsilon_{345},k_2)(\epsilon_{61},k_2)-(\epsilon_2,k_{61})(\epsilon_{34},k_2)(\epsilon_1,k_2)\right.\\
&- \left.\frac{\langle 2\,5\rangle}{\langle 2\,6\rangle}(\epsilon_2,k_{61})(\epsilon_{345},k_2)+\frac{\langle 2\,6\rangle}{\langle 2\,5\rangle}(\epsilon_{34},k_2)(\epsilon_{61},k_2)\right]\\
&+ \frac{(s_{34}+s_{234})}{6(s_{34}-s_{234})^2}\left[\frac{\langle 2\,6\rangle}{\langle 2\,5\rangle}(\epsilon_{34},k_2)(\epsilon_{61},k_2)\,(\epsilon_2,k_{34})\right.\\
&- \left.\frac{1}{2}(\epsilon_2,k_{34})(\epsilon_{34},k_2)(\epsilon_1,k_2)-(\epsilon_2,k_{34})(\epsilon_{34},k_2)(\epsilon_{561},k_2)\right]
\end{aligned}
$$

$$\times \frac{1}{2\,t_{345}}\left(\frac{\langle 1^-|\,(3+4)\,|5^-\rangle\,\langle 1\,3\rangle\langle 2\,6\rangle\langle 3\,5\rangle}{\langle 3\,6\rangle}-\frac{\langle 1^-|\,(2+6)(3+5)\,|2^+\rangle\,\langle 1\,2\rangle\langle 3\,6\rangle}{\langle 2\,6\rangle}\right) \tag{B4}$$

For $A_{n;1}^{\mathcal{N}=0}(1^-,2^+,3^+,4^-,5^+,6^+)$ we can make use of the symmetry to write the rational remainder as

$$\widehat{R}_6(1,4)=\widehat{R}_6^a(1,4)+\widehat{R}_6^a(1,4)\Big|_{flip} \tag{B2}$$

where

$$X(1,2,3,4,5,6)\Big|_{\rm flip}\equiv X(1,6,5,4,3,2) \tag{B3}$$

and *Beware of and make sure the* $2/9A^{tree}$ *is right. Also make sure subtraction of boundary of cut is properly included!!!*

$$\begin{aligned}
&\widehat{R}_6^a(1,4)\\
&=-\frac{8\langle 1\,3\rangle\langle 1\,4\rangle^3}{9\langle 1\,2\rangle\langle 1\,6\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 3\,5\rangle\langle 5\,6\rangle}-\frac{(\langle 1\,4\rangle[4\,3]+2\langle 1\,5\rangle[5\,3])\,\langle 1^-|\,(4+5)\,|3^-\rangle\,\langle 1\,5\rangle^2[3\,5]}{6\langle 1\,2\rangle\langle 1\,6\rangle\langle 2\,5\rangle\langle 3\,5\rangle\langle 5\,6\rangle[3\,4]^2t_{345}}\\
&+\frac{\langle 2^-|\,(1+4)\,|6^-\rangle\,\langle 5^-|\,(1+4)\,|6^-\rangle\,\langle 1\,5\rangle^2[5\,6]t_{146}}{2s_{16}\,\langle 3^-|\,(1+4)\,|6^-\rangle\,\langle 5^-|\,(2+3)\,|4^-\rangle\,\langle 2\,3\rangle\langle 2\,5\rangle^2\langle 5\,6\rangle[4\,6]}\\
&-\frac{[3\,5]^3[4\,6]^3}{3\,\langle 2^-|\,(3+5)\,|4^-\rangle^2\,[1\,4][1\,6][3\,4][4\,5]}+\frac{\langle 1^-|\,(2+4)\,|3^-\rangle^3\,[2\,4]}{3\,\langle 5^-|\,(2+3)\,|4^-\rangle\,\langle 1\,6\rangle\langle 2\,3\rangle\langle 5\,6\rangle[3\,4]^2t_{234}}\\
&+\frac{\langle 1^-|\,(2+4)\,|3^-\rangle\,\langle 1\,2\rangle[2\,3](\langle 1^-|\,(2+4)\,|3^-\rangle+\langle 1\,2\rangle[2\,3])}{6\langle 1\,6\rangle\langle 2\,3\rangle\langle 2\,5\rangle\langle 5\,6\rangle[3\,4]^2t_{234}}\\
&+\frac{\langle 1^-|\,(2+4)\,|3^-\rangle^2\,\langle 1\,2\rangle\langle 1\,5\rangle[2\,3]}{2\,\langle 1^-|\,(2+3)\,|4^-\rangle\,\langle 1\,6\rangle\langle 2\,5\rangle^2\langle 5\,6\rangle[3\,4]t_{234}}\\
&-\frac{\langle 5^-|\,(1+4)\,|6^-\rangle\,\langle 1\,4\rangle\langle 1\,5\rangle[5\,6]t_{146}}{6s_{16}\,\langle 3^-|\,(1+4)\,|6^-\rangle\,\langle 5^-|\,(2+3)\,|4^-\rangle\,\langle 2\,3\rangle\langle 2\,5\rangle\langle 5\,6\rangle}\\
&-\frac{\langle 1\,2\rangle\langle 1\,4\rangle^2[2\,3]}{6\langle 1\,6\rangle\langle 2\,3\rangle\langle 2\,5\rangle\langle 3\,4\rangle\langle 5\,6\rangle[3\,4]}\left(1+\frac{3\langle 1\,5\rangle\langle 2\,3\rangle}{\langle 1\,3\rangle\langle 2\,5\rangle}\right)\\
&+\frac{\langle 2^-|\,(1+4)\,|6^-\rangle^3\,\langle 3^-|\,(2+5)\,|4^-\rangle\,[2\,3][4\,6]}{3\,\langle 2^-|\,(3+5)\,|4^-\rangle^2\,\langle 5^-|\,(1+4)\,|6^-\rangle\,\langle 5^-|\,(2+3)\,|4^-\rangle\,\langle 2\,3\rangle^2[1\,4][1\,6]}\\
&+\frac{\langle 1\,4\rangle^2\langle 1\,5\rangle^2[3\,5]}{6\langle 1\,2\rangle\langle 1\,6\rangle\langle 2\,5\rangle\langle 3\,4\rangle\langle 3\,5\rangle\langle 5\,6\rangle[3\,4]}\left(1+\frac{3\langle 1\,2\rangle\langle 3\,5\rangle}{\langle 1\,3\rangle\langle 2\,5\rangle}\right)\\
&+\frac{\langle 1\,4\rangle\langle 1\,5\rangle[2\,3][5\,6]}{6\,\langle 1^-|\,(3+4)\,|2^-\rangle\,\langle 2\,5\rangle\langle 5\,6\rangle t_{234}}\left(\frac{\langle 1\,4\rangle}{\langle 2\,3\rangle}+\frac{3\,\langle 1^-|\,(2+4)\,|3^-\rangle\,\langle 1\,5\rangle}{\langle 1^-|\,(2+3)\,|4^-\rangle\,\langle 2\,5\rangle}\right)\\
&-\frac{\langle 1^-|\,(3+5)\,|4^-\rangle^4\,[3\,5]^3}{6\,\langle 2^-|\,(3+5)\,|4^-\rangle\,\langle 6^-|\,(3+5)\,|4^-\rangle\,\langle 1\,2\rangle\langle 1\,6\rangle\langle 3\,5\rangle[3\,4]^2[4\,5]^2t_{345}}\\
&+\frac{\langle 1^-|\,(3+5)\,|4^-\rangle^4\,\langle 2\,6\rangle[2\,6][3\,5]^4}{6\,\langle 1^-|\,(3+4)\,|5^-\rangle\,\langle 1^-|\,(4+5)\,|3^-\rangle\,\langle 2^-|\,(3+5)\,|4^-\rangle^2\,\langle 6^-|\,(3+5)\,|4^-\rangle^2\,[3\,4][4\,5]t_{345}}\\
&+\frac{\langle 1^-|\,(2+6)(3+5)\,|6^+\rangle^2\,\langle 1\,6\rangle[2\,6][3\,5]}{3\,\langle 1^-|\,(3+4)\,|5^-\rangle\,\langle 1^-|\,(4+5)\,|3^-\rangle\,\langle 6^-|\,(3+5)\,|4^-\rangle\,\langle 2\,6\rangle\langle 3\,6\rangle\langle 5\,6\rangle t_{345}}\\
&\quad\times\left(\frac{\langle 1\,4\rangle}{2\langle 3\,5\rangle}-\frac{\langle 1\,6\rangle[3\,5]}{\langle 6^-|\,(3+5)\,|4^-\rangle}\right)\\
&-\frac{\langle 5^-|\,(2+3)\,|6^-\rangle\,[5\,6]t_{146}^2([1\,4]\langle 1\,2\rangle\langle 1\,5\rangle+\langle 2\,5\rangle\langle 1\,6\rangle[4\,6])}{3\,\langle 2^-|\,(3+5)\,|4^-\rangle\,\langle 3^-|\,(1+4)\,|6^-\rangle\,\langle 5^-|\,(2+3)\,|4^-\rangle\,\langle 1\,6\rangle\langle 2\,3\rangle\langle 5\,6\rangle[1\,6]\langle 2\,5\rangle[1\,4]}\\
&-\frac{\langle 1\,5\rangle^2[2\,3]^2[5\,6]}{6\,\langle 1^-|\,(3+4)\,|2^-\rangle\,\langle 5^-|\,(2+3)\,|4^-\rangle\,\langle 2\,5\rangle\langle 5\,6\rangle t_{234}}\\
&\quad\times\left(3\langle 1\,4\rangle+\frac{3\,\langle 1^-|\,(2+4)\,|3^-\rangle\,\langle 1\,5\rangle\langle 2\,3\rangle}{\langle 1^-|\,(2+3)\,|4^-\rangle\,\langle 2\,5\rangle}-\frac{2\langle 1\,5\rangle\langle 2\,6\rangle[2\,3]}{\langle 5\,6\rangle[3\,4]}\right)\\
&+\frac{\langle 1\,4\rangle^3\langle 1\,5\rangle[2\,5]}{6\,\langle 1^-|\,(3+4)\,|2^-\rangle\,\langle 3^-|\,(1+4)\,|6^-\rangle\,\langle 1\,3\rangle\langle 1\,6\rangle^2\langle 2\,3\rangle\langle 2\,5\rangle^2\langle 3\,4\rangle\langle 5\,6\rangle}\\
&\quad\times\Big((\langle 1^-|\,6(2+5)\,|3^+\rangle+\langle 1^-|\,(5+6)2\,|3^+\rangle)(\langle 1\,3\rangle\langle 2\,5\rangle-3\langle 1\,5\rangle\langle 2\,3\rangle)\\
&\qquad-3\langle 1\,2\rangle\langle 1\,5\rangle\langle 2\,3\rangle\langle 3\,5\rangle[2\,5]\Big)\\
&+\frac{\langle 1\,4\rangle^3\langle 3\,6\rangle[2\,6]}{6\,\langle 1^-|\,(3+4)\,|5^-\rangle\,\langle 3^-|\,(1+4)\,|2^-\rangle\,\langle 1\,2\rangle\langle 1\,3\rangle\langle 2\,6\rangle^2\langle 3\,4\rangle\langle 3\,5\rangle^2\langle 5\,6\rangle}\\
&\quad\times\Big(2(\langle 1^-|\,2(5+6)\,|3^+\rangle+\langle 1^-|\,(2+6)5\,|3^+\rangle)\langle 1\,3\rangle\langle 2\,6\rangle\\
&\qquad-3(2\,\langle 1^-|\,2(5+6)\,|3^+\rangle+\langle 1^-|\,65\,|3^+\rangle)\langle 1\,6\rangle\langle 2\,3\rangle\Big)\\
&-\frac{\langle 1^-|\,(4+5)\,|3^-\rangle\,\langle 1\,5\rangle^2[3\,5]}{2\,\langle 1^-|\,(3+5)\,|4^-\rangle\,\langle 1\,2\rangle\langle 1\,6\rangle\langle 2\,3\rangle\langle 2\,5\rangle\langle 3\,5\rangle\langle 5\,6\rangle[3\,4]t_{345}}\\
&\quad\times\left(\frac{\langle 1^-|\,(2+6)(3+5)\,|2^+\rangle\,\langle 1\,2\rangle\langle 3\,5\rangle}{\langle 2\,5\rangle}-\langle 1^-|\,(3+4)\,|5^-\rangle\,\langle 1\,3\rangle\langle 2\,5\rangle\right)\\
&+\frac{\langle 2^-|\,(1+4)\,|6^-\rangle\,t_{146}^2}{3s_{16}\,\langle 2^-|\,(3+5)\,|4^-\rangle^2\,\langle 3^-|\,(1+4)\,|6^-\rangle\,\langle 2\,3\rangle\langle 2\,5\rangle\langle 2\,6\rangle[1\,4]}\\
&\quad\times\left(\frac{\langle 1\,6\rangle\langle 2\,5\rangle\langle 2\,6\rangle[4\,6]^2[5\,6]}{\langle 5^-|\,(2+3)\,|4^-\rangle}-\frac{\langle 2^-|\,(1+4)\,|6^-\rangle\,\langle 1\,2\rangle^2[1\,4][2\,6]}{\langle 5^-|\,(2+3)\,|6^-\rangle}\right)\\
&+\frac{\langle 2^-|\,(1+4)\,|6^-\rangle^2\,\langle 1\,2\rangle[2\,6]t_{146}}{2s_{16}\,\langle 2^-|\,(3+5)\,|4^-\rangle\,\langle 3^-|\,(1+4)\,|6^-\rangle\,\langle 5^-|\,(1+4)\,|6^-\rangle\,\langle 2\,3\rangle\langle 2\,5\rangle\langle 2\,6\rangle\langle 5\,6\rangle[4\,6]}\\
&\quad\times\left(\frac{\langle 5^-|\,(1+4)\,|6^-\rangle\,\langle 1\,5\rangle\langle 2\,6\rangle}{\langle 2\,5\rangle}-\frac{\langle 6^-|\,(1+4)\,|6^-\rangle\,\langle 1\,6\rangle\langle 2\,5\rangle}{\langle 2\,6\rangle}+\frac{\langle 1\,4\rangle\langle 5\,6\rangle[4\,6]}{3}\right)\\
&+\frac{\langle 1^-|\,(2+6)(3+5)\,|6^+\rangle^2\,\langle 1\,6\rangle[2\,6][3\,5]}{\langle 1^-|\,(3+4)\,|5^-\rangle\,\langle 1^-|\,(3+5)\,|4^-\rangle\,\langle 1^-|\,(4+5)\,|3^-\rangle\,\langle 6^-|\,(3+5)\,|4^-\rangle\,\langle 2\,3\rangle\langle 2\,6\rangle\langle 3\,5\rangle\langle 3\,6\rangle\langle 5\,6\rangle}
\end{aligned}$$

Mathematica 5.1 - [Untitled-1 *]

File Edit Cell Format Input Kernel Find Window Help

Untitled-1 *

In[72]:= `p00 = %70`

Out[72]=

```
2502211875667893213827355686545267949740267678120686279280535643969788421343795811361 5
/
1168075627225659898464419664108806338586752881199234982762109893273398374565839287 4
```

In[73]:=

```
p01 =
 1883542439871892835270646119638452293261666916725576628284162843426789158291318033355
  /
  4492598566252538071016998708110793609949049543073980702931191897205378363714766 49 -
 2379093105536000 0 / (518397975010773 * e)
```

Out[73]=

$$\frac{1883542439871892835270646119638452293261666916725576628284162843426789158291318033355}{449259856625253807101699870811079360994904954307398070293119189720537836371476649} - \frac{23790931055360000}{518397975010773\,e}$$

In[74]:= `pp = p01 - 2 p00`

Out[74]=

$$-\frac{47581862110720000}{518397975010773} - \frac{23790931055360000}{518397975010773\,e}$$

In[75]:= `Factor[%]`

Out[75]=

$$-\frac{23790931055360000\,(1 + 2\,e)}{518397975010773\,e}$$

200%

开始 Squirre... 5.1 大智慧... 中国银... 分析家 ... 航空证... 5 记事本 Mathema... Mathema... LView P... 81% 11:09

NMHV: $A_6(1^-2^-3^+4^-5^+6^+)$

Some terms are:

$$
\begin{aligned}
R_1 &= I_3^{3m}(k_1,\epsilon_3,\epsilon_4) + (\epsilon_4,k_5)\, I_3^{3m}(k_1,\epsilon_3) \\
&- \frac{1}{2}\left[\tilde{I}_3^{2m(4)}(\epsilon_4,k_1,\epsilon_3) + (s_{56}-s_{234})\,\tilde{I}_3^{2m(4)}(\epsilon_4,\epsilon_3)\right] \\
&- \frac{1}{2}\left[I_3^{2m(3)}(\epsilon_3,\epsilon_4,k_{12}) + s_{12}\, I_3^{2m(3)}(\epsilon_3,\epsilon_4)\right] \\
&+ \frac{1}{36}\, s_{34}\,(\epsilon_{34},k_{12}) - \frac{1}{36}((\epsilon_3,k_2)(\epsilon_4,k_3) - 2s_{23}(\epsilon_3,\epsilon_4)) \\
&- \frac{1}{72}((\epsilon_3,k_{12})(\epsilon_4,k_{56}) + 2s_{123}(\epsilon_3,\epsilon_4))
\end{aligned}
$$

$$
\begin{aligned}
R_2 \quad = \quad & -\langle 4\,3\rangle[3\,2]\big(I_4^{2mh(2)}(\lambda_2\tilde{\lambda}_1,\lambda_6\tilde{\lambda}_3,\epsilon_{61}) - I_4^{1m(2)}(\lambda_2\tilde{\lambda}_1,\lambda_6\tilde{\lambda}_3,\epsilon_{61})\big) \\
+ \quad & \langle 5\,4\rangle[4\,3]\,\big(I_4^{1m(3)}(\lambda_4\tilde{\lambda}_1,\lambda_6\tilde{\lambda}_5,\epsilon_{61}) - I_4^{2mh(4)}(\lambda_4\tilde{\lambda}_1,\lambda_6\tilde{\lambda}_5,\epsilon_{61}\big) \\
+ \quad & I_3^{2m(3)}(\lambda_6\tilde{\lambda}_3,\epsilon_{61},\lambda_4\tilde{\lambda}_1) - \langle 6|k_{45}|3]\, I_3^{2m(3)}(\lambda_4\tilde{\lambda}_1,\epsilon_{61}) \\
- \quad & \Big((\epsilon_{61},k_{45}) + \frac{1}{2}(\epsilon_6,\epsilon_1)\Big)\, I_3^{2m(3)}(\lambda_6\tilde{\lambda}_3,\lambda_4\tilde{\lambda}_1) \\
+ \quad & \tilde{I}_3^{2m(4)}(\lambda_4\tilde{\lambda}_1,\lambda_6\tilde{\lambda}_3,\epsilon_{61}) + \langle 4|k_{23}|1]\, \tilde{I}_3^{2m(4)}(\lambda_6\tilde{\lambda}_3,\epsilon_{61}) \\
+ \quad & \Big((\epsilon_{61},k_{23}) - \frac{1}{2}(\epsilon_6,\epsilon_1)\Big)\, \tilde{I}_3^{2m(4)}(\lambda_6\tilde{\lambda}_3,\lambda_4\tilde{\lambda}_1) \\
- \quad & I_3^{3m}(\lambda_4\tilde{\lambda}_1,\epsilon_{61},\lambda_6\tilde{\lambda}_3) - \Big((\epsilon_{61},k_{23}) - \frac{1}{2}(\epsilon_6,\epsilon_1)\Big)\, I_3^{3m}(\lambda_4\tilde{\lambda}_1,\lambda_6\tilde{\lambda}_3) \\
+ \quad & \frac{7}{18}\, s_{34}(\tilde{\epsilon}_{34},\epsilon_{61}) + \frac{1}{4}\,\langle 6\,4\rangle\,[1\,3]\,(\epsilon_{61},k_3 - k_4)
\end{aligned}
$$

(Compact) formula for rational part of tensor box integral – 3m and 4m cases

$$I_{4,m}^D = \int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}} \frac{\prod_{i=1}^{m}(\epsilon_i, p)}{p^2(p+K_1)^2(p+K_2)^2(p+K_3)^2}$$

$K_{1,2,3}$ **and** $K \equiv \epsilon_{\mu\nu\rho\sigma} K_1^\nu K_2^\rho K_3^\sigma$**.** $\det K_i \cdot K_j \propto K^2$.

Expanding ϵ_i **in terms of** K_i **and** K**:**

$$\epsilon_i = \sum_{j=1}^{3} a_{ij}\, K_j + c_i K$$

$$(K_i, p) = (p + K_i)^2 - p^2 - K_i^2$$

An example: linear box integral

$$I_{4,1}^D = \int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}} \frac{(\epsilon, p)}{p^2 (p+K_1)^2 (p+K_2)^2 (p+K_3)^2}$$

$$\to \Big(-\sum_{i=1}^{3} a_i K_i^2 \Big) \, I_4^D[1] + \sum_{i=1}^{3} a_i \, I_3^{D(i)}[1] - (a_1 + a_2 + a_3) \, I_3^{D(0)}[1]$$

We note:

$$-\sum_{i=1}^{3} a_i K_i^2 = \frac{1}{2} \left((\epsilon, p^+) + (\epsilon, p^-) \right)$$

with $p^2 = 0$ **and** $(p + K_i)^2 = 0$**.** $(K, p^+) = -(K, p^-)$

(Quadruple cuts to compute box coefficients: BCF, hep-th/0412103).

$$\int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}}\,\frac{(K,p)^2}{p^2(p+K_1)^2(p+K_2)^2(p+K_3)^2}$$
$$= -\frac{1}{2}\,K^2\,I_4^{D+2}[1] \to 0.$$

$$\int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}}\,\frac{(K,p)^4}{p^2(p+K_1)^2(p+K_2)^2(p+K_3)^2}$$
$$= \frac{3}{4}\,(K^2)^2\,I_4^{D+4}[1] \to \frac{3}{4} \times \frac{5}{18}\,(K^2)^2.$$

Compact formula for triangle integral?

$$I_{3,m}^{D} = \int \frac{\mathrm{d}^D p}{i\,\pi^{D/2}} \, \frac{\prod_{i=1}^{m} (\epsilon_i, p)}{p^2 (p+K_1)^2 (p+K_2)^2}$$

Expanding ϵ_i in terms of:

$$K_1, K_2, l, \bar{l},$$

$$(K_i, l) = 0, \quad l^2 = 0, \quad l = \lambda\,\bar{\mu},$$

$$(K_i, \bar{l}) = 0, \quad \bar{l}^2 = 0, \quad \bar{l} = \mu\,\bar{\lambda}.$$

An example: linear triangle integral

$$
\begin{aligned}
I^D_{3,1} &= -(a_1 K_1^2 + a_2 K_2^2)\, I_3^D[1] - (a_1 + a_2)\, I_2^{D(0)}[1] \\
&\quad + a_1\, I_2^{D(1)}[1] + a_2\, I_2^{D(2)}[1],
\end{aligned}
$$

by expanding $\epsilon = a_1 K_1 + a_2 K_2 + \bar{c}\, l + c \bar{l}$.

See recent paper: G. Ossola, C. Papadopoulos and R. Pittau, hep-ph/0609007, eq. (2.4).

5. 展望

- **Seems feasible for $n = 7, 6, \cdots$: both cut-part (triangle and bubble coefficients) and rational part**
  - **high-point tensor integrals: direct reduction by inserting "spinor-string" (BDK 98) and keeping only $n, n-1, n-2$ and $n-3$ tensors (BDDK theorem)**
  - **box and triangle integrals: red. by expanding ϵ_i. (See also: Binoth et. al., hep-th/0609054)**
- **Attacking the wish lists.**

What needs to be done at NLO?

Experimenters to theorists: "Please calculate the following at NLO"

Run II Monte Carlo Workshop, April 2001

Single boson	Diboson	Triboson	Heavy flavour
$W + \leq 5j$	$WW + \leq 5j$	$WWW + \leq 3j$	$t\bar{t} + \leq 3j$
$W + b\bar{b} + \leq 3j$	$WW + b\bar{b} + \leq 3j$	$WWW + b\bar{b} + \leq 3j$	$t\bar{t} + \gamma + \leq 2j$
$W + c\bar{c} + \leq 3j$	$WW + c\bar{c} + \leq 3j$	$WWW + \gamma\gamma + \leq 3j$	$t\bar{t} + W + \leq 2j$
$Z + \leq 5j$	$ZZ + \leq 5j$	$Z\gamma\gamma + \leq 3j$	$t\bar{t} + Z + \leq 2j$
$Z + b\bar{b} + \leq 3j$	$ZZ + b\bar{b} + \leq 3j$	$WZZ + \leq 3j$	$t\bar{t} + H + \leq 2j$
$Z + c\bar{c} + \leq 3j$	$ZZ + c\bar{c} + \leq 3j$	$ZZZ + \leq 3j$	$t\bar{b} + \leq 2j$
$\gamma + \leq 5j$	$\gamma\gamma + \leq 5j$		$b\bar{b} + \leq 3j$
$\gamma + b\bar{b} + \leq 3j$	$\gamma\gamma + b\bar{b} + \leq 3j$		
$\gamma + c\bar{c} + \leq 3j$	$\gamma\gamma + c\bar{c} + \leq 3j$		
	$WZ + \leq 5j$		
	$WZ + b\bar{b} + \leq 3j$		
	$WZ + c\bar{c} + \leq 3j$		
	$W\gamma + \leq 3j$		
	$Z\gamma + \leq 3j$		

Theorists to experimenters:

"In your dreams"

More Realistic Experimenter's Wish List

Les Houches 2005

process ($V \in \{Z, W, \gamma\}$)	background to
1. $pp \to V\,V\,\text{jet}$	$t\bar{t}H$, new physics
2. $pp \to H + 2\,\text{jets}$	H production by vector boson fusion (VBF)
3. $pp \to t\bar{t}\,b\bar{b}$	$t\bar{t}H$
4. $pp \to t\bar{t} + 2\,\text{jets}$	$t\bar{t}H$
5. $pp \to V\,V\,b\bar{b}$	VBF$\to H \to VV$, $t\bar{t}H$, new physics
6. $pp \to V\,V + 2\,\text{jets}$	VBF$\to H \to VV$
7. $pp \to V + 3\,\text{jets}$	various new physics signatures
8. $pp \to V\,V\,V$	SUSY trilepton

Bold action is required even for this